Chern Simons theory on a lattice and a new description of 3-manifolds invariants.

E.Buffenoir*,

Centre de Physique Theorique Ecole Polytechnique
91128 Palaiseau Cedex
France

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Abstract

A new approach to the quantization of Chern-Simons theory has been developed in recent papers [1, 2, 3, 4]. It uses a "simulation" of the moduli space of flat connections modulo the gauge group which reveals to be related to a lattice gauge theory based on a quantum group. After a generalization of the formalism of q-deformed gauge theory to the case of root of unity, we compute explicitely the correlation functions associated to Wilson loops (and more generally to graphs) on a surface with punctures, which are the interesting quantity in the study of moduli space. We then give a new description of Chern-Simons three manifolds invariants based on a description in terms of the mapping class group of a surface. At last we introduce a three dimensional lattice gauge theory based on a quantum group which is a lattice regularization of Chern-Simons theory.

1 Introduction

This paper is the third part of a study of combinatorial quantization of Chern-Simons theory[2, 3]. Several ideas developed here are in fact products of those introduced by V.V.Fock and A.A.Rosly in their study of Poisson structures on the moduli space of flat connections[1]. To understand the motivation of these papers we must recall some general facts about 3D Chern-Simons theory. Chern-Simons theory is a gauge theory in 3 dimensions defined by the action principle

$$\mathcal{S}_{CS} = \frac{k}{4\pi} tr(\int_{\mathcal{M}} \mathcal{A} d\mathcal{A} + \frac{2}{3}\mathcal{A}^3) \qquad (1)$$

*e-mail: buffenoi@orphee.polytechnique.fr

where $\mathcal{M}$ is a 3-manifold, k a positive integer and $\mathcal{A}$ a connection associated to a semisimple Lie algebra $\mathcal{G}$. If we first suppose that the manifold locally looks like a cylinder $\Sigma \times R$, R considered as the the time direction, we can consider the Chern-Simons theory in an Hamiltonian point of view. We will denote by A the two space-components of the gauge field taken to be the dynamical variables of the theory, and the time component A_0 will become a Lagrange multiplier. With these notations the action can be written:

$$\mathcal{S}_{CS} = \frac{k}{4\pi} tr(\int_{\mathcal{M}} (-A\partial_0 A + 2A_0 F) dt). \tag{2}$$

The first term gives the Poisson structure:

$$\{A_i^a(x,y), A_j^b(x',y')\} = \frac{-2\pi}{k} \delta^{ab} \epsilon_{ij} \delta^{(2)}((x,y),(x',y')). \tag{3}$$

The hamiltonian is a combination of constraints and the second term imposes as a constraint that the curvature of the connection A is zero:

$$F = dA + A^2 = 0. \tag{4}$$

Computing the Poisson brackets of the constraints we obtain that they are first class:

$$\{F^a(x,y), F^b(x',y')\} = \frac{2\pi}{k} f_c^{ab} F^c(x,y) \delta^{(2)}((x,y)(x',y')). \tag{5}$$

where the f_c^{ab} are the structure constants of the Lie algebra $\mathcal{G}$. The constraints (4) generate the infinitesimal gauge transformations of the gauge field then the phase space of the hamiltonian Chern-Simons theory is the space of flat connections modulo gauge transformations.
Quantizing the latter Poisson structure of gauge fields in the usual way,

$$[A_i^a(x,y), A_j^b(x',y')] = \frac{-2\pi}{k} \delta^{ab} \epsilon_{ij} \delta^{(2)}((x,y)(x',y')) \tag{6}$$

we are led to work with an infinite dimensional algebra, observables becoming functionals over the elements of this algebra. The idea of V.V.Fock and A.A.Rosly is completely different.Let us briefly describe it.
The space of flat connections modulo the gauge group having only a finite number of freedom degrees, we reduce the connection to live on a graph encoding the topology of the surface and equip this "graph connection" with a Poisson structure such that the Poisson structure induced on the gauge invariant observables is compatible with that induced from the usual symplectic structure. We obtain a "simulation" of hamiltonian Chern-Simons theory in the sense that the operator algebra derived from both descriptions are the same.

We will consider a graph dividing the surface into contractile plaquettes. This graph will be equiped with an additionnal structure of "ciliated fat graph",

i.e. a graph with a linear order between adjacent links at each vertex. Let x, y be neighbour vertices of the graph, we will denote by $U_{[x,y]}$ the parallel transport operator associated to the link and the connection. The gauge group acts in the usual way on this object:

$$U_{[x,y]} \rightarrow g_x U_{[x,y]} g_y^{-1} \tag{7}$$

We can define a Lie Poisson structure on such objects [1] owning the following fascinating property:
the space of flat graph connections modulo graph gauge group is Lie Poisson isomorphic to the space of flat connections modulo gauge group on the surface.

The problem has been reduced to a finite dimensional problem by limiting the gauge group to act at a finite number of sites.
The quantization of such a Lie Poisson structure leads us to an exchange algebra on which acts a quantum group. The object of [2, 4] was to define this algebraic structure.

In a second time we found a projector in this algebra imposing "a posteriori" the flatness condition, the result was then a two dimensional lattice gauge theory based on a quantum group. The correlation functions associated to gauge invariant objects in this theory being related to expectation values in Chern-Simons theory.

In our second paper [3] we further investigated the algebra of gauge invariant elements, particularily the algebra associated to loops. Our aim was to describe a new approach to knots invariants, showing in a well defined framework the relation between Reshetikhin-Turaev invariants and Chern-Simons theory.

The aim of the following paper is double:

- to investigate further the computation of the correlation functions of our theory on any surface and construct the derived invariants associated to the mapping class group, and establish a new description of three manifold invariants using a generalization of our previous construction to the case of roots of unity

- to build up a three dimensional lattice q-gauge theory associated to triangulations of any 3-manifolds which will describe a well defined finite path integral formula for Chern-Simons theory and the way to compute any correlation functions of this theory.
  This work is a tentativ to revisit the work of E.Witten [5] with a well

defined formalism allowing a lot of new computations and specially computations of invariants associated to intersecting loops in Chern-Simons theory.

2 Lattice gauge theory based on a quantum group

2.1 Quantum groups and exchange algebras associated to fat graphs

In this chapter, after a brief summary of the results of [6] [7], we will further develop the notion of quantum group at root of unity in the dual version and then, using this construction, generalize the results on gauge fields algebra developed in [2, 3] in a way quite different from that described in [4]. We will consider a Hopf algebra $(\mathcal{A}, m, 1, \Delta, S, \epsilon)$. To simplify we will take $\mathcal{A} = \mathcal{U}_q(sl_2)$ with q being a complex number different from ± 1. As usual we will refer to $\mathbf{R}$ as the universal $R-$matrix associated to $\mathcal{A}$ (we will often write $\mathbf{R} = \sum_i a_i \otimes b_i$), u the element defined by $u = \sum_i S(a_i) b_i$ verifying the usual properties and v the ribbon element defined by $v^2 = uS(u)$ (for details see [8, 9, 10]).
Depending on whether q is a root of unity or not, the representation theory of $\mathcal{A}$ is completely different. We will denote by $Irr(\mathcal{A})$ the set of equivalence classes of finite dimensional irreducible representations of $\mathcal{A}$. In each class $\dot{\alpha}$ we will pick out a representativ α and, like often in physics, denote equivalently by α or $V_\alpha$the representation space associated to α. We will denote by $\bar{\alpha}$ (resp. $\tilde{\alpha}$) the right (resp. the left) contragredient representation build up from the antipode (resp. the inverse of the antipode) by $\bar{\alpha} = {}^t\alpha \circ S$ and 0 the one dimensional representation associated to ϵ. The tensor product of two representations is defined by the coproduct Δ. If q is a root of unity the decomposition of the tensor product of two irreducible representations can involve indecomposable representations (i.e representations which are not irreducible but cannot ye! t be decomposed in a direct sum of stable $\mathcal{A}-$modules). We are able to introduce $\Psi^{\gamma,m}_{\alpha\beta}$ and $\Phi^{\alpha\beta}_{\gamma,m}$ respectively projection of the tensor product of α and β on the m-th isotypic component γ and the inclusion of γ in the tensor product $\alpha \otimes \beta$. Using this notation we will make one more restriction between "physical" representations verifying $\Psi^0_{\alpha\bar{\alpha}} \Phi^{\alpha\bar{\alpha}}_0 \neq 0$ and the other representations, we will denote by $Phys(\mathcal{A})$ this subset of $Irr(\mathcal{A})$. We will introduce a new tensor product between elements of $Phys(\mathcal{A})$ simply realizing a truncation of the previous one, defined by:

$$\alpha \otimes \beta = \bigoplus_{\gamma \in Phys(\mathcal{A})} N^{\alpha\beta}_\gamma \gamma. \tag{8}$$

N is the fusion matrix of $\mathcal{A}$ and we will also use the notation $\delta(\alpha\beta\gamma)$ to be equal to 1 or 0 depending on whether γ occurs or not in the decomposition of the tensor product $\alpha \otimes \beta$. If q is generic the tensor product of two elements of $Irr(A)$ can be decomposed in a direct sum of elements of $Irr(A)$. Moreover

all irreducible representations are "physical", so we do not have to change the tensor product in this case. We will associate new projection and inclusion operators $\psi^{\gamma,m}_{\alpha\beta}$ and $\phi^{\alpha\beta}_{\gamma,m}$ build up from the truncated tensor product. We will also use the $6-j$ notation $\left\{\begin{matrix} \alpha & \beta & \gamma \\ \delta & \mu & \nu \end{matrix}\right\}_q$ defined in the usual way from projection and inclusion operators (see [11] for definitions and properties).
We will use the following notation replacing the coproduct by a truncated coproduct for any element ξ of the algebra $\mathcal{A}$:

$$\overset{\alpha\otimes\beta}{\xi} = \sum_{\gamma\in Phys(\mathcal{A})} \phi^{\alpha\beta}_{\gamma,m} \overset{\gamma}{\xi} \psi^{\gamma,m}_{\alpha\beta}. \tag{9}$$

The antipode and counity maps do not change through truncation and we will again have:

$$\overset{\alpha}{S}(\xi) = {}^t\overset{\bar{\alpha}}{\xi} \tag{10}$$

The first trivial properties of the projection and inclusion operators are:

$$\psi^{\gamma',m'}_{\alpha\beta}\phi^{\alpha\beta}_{\gamma,m} = \delta_{m,m'}\delta_{\gamma,\gamma'}\delta(\alpha\beta\gamma)id_\gamma \tag{11}$$

$$\sum_{\gamma\in Phys(\mathcal{A}),m} \phi^{\alpha\beta}_{\gamma,m}\psi^{\gamma,m}_{\alpha\beta} = \overset{\alpha\otimes\beta}{\mathbf{1}} \tag{12}$$

$$\phi^{\alpha 0}_{\beta} = \phi^{0\alpha}_{\beta} = \psi^{\beta}_{\alpha 0} = \phi^{\beta}_{0\alpha} = \delta_{\alpha\beta} id_\alpha. \tag{13}$$

The essential fact is that when the truncation is not trivial (i.e in the root of unity case) the representations $((\alpha\otimes\beta)\otimes\gamma)$ and $(\alpha\otimes(\beta\otimes\gamma))$ are no more equal but are equivalent, the intertwiner map between them being $\overset{\alpha\beta\gamma}{\Theta}$ defined by:

$$\overset{\alpha\beta\gamma}{\Theta} = \sum_{\delta,\nu,\mu\in Phys(\mathcal{A})} \left\{\begin{matrix} \gamma & \beta & \delta \\ \alpha & \nu & \mu \end{matrix}\right\}_q \phi^{\beta\gamma}_{\delta}\phi^{\alpha\delta}_{\nu}\psi^{\mu\gamma}_{\nu}\psi^{\alpha\beta}_{\mu} \tag{14}$$

(where we have omited the multiplicities to simplify the notation, it will often be the case in the following). We will denote by $\overset{\alpha\beta\gamma}{\Theta^{-1}}$ its quasi-inverse.

We will often use the notation $\overset{\alpha\beta\gamma}{\Theta}_{123} = \sum_i \overset{\alpha}{\theta^{(1)}_i} \otimes \overset{\beta}{\theta^{(2)}_i} \otimes \overset{\gamma}{\theta^{(3)}_i}$, and the coproduct notations $\overset{(\alpha\otimes\beta)\gamma\delta}{\Theta}_{1234} = \sum_i \overset{\alpha}{\theta^{(11)}_i} \otimes \overset{\beta}{\theta^{(12)}_i} \otimes \overset{\gamma}{\theta^{(2)}_i} \otimes \overset{\delta}{\theta^{(3)}_i}...$

In the case where q is generic $\overset{\alpha\beta\gamma}{\Theta}$ is simply the identity but more generally it is possible to collect some interesting properties in the root of unity case. Using the pentagonal identity and other trivial identities on $6-j$ symbols we

can verify:

$$\overset{\alpha\beta(\gamma\otimes\delta)}{\Theta}\ \overset{(\alpha\otimes\beta)\gamma\delta}{\Theta} = (\overset{\alpha}{\mathbf{1}}\otimes\overset{\beta\gamma\delta}{\Theta})\ \overset{\alpha(\beta\otimes\gamma)\delta}{\Theta}\ (\overset{\alpha\beta\gamma}{\Theta}\otimes\overset{\delta}{\mathbf{1}}) \tag{15}$$

$$\overset{0\alpha\beta}{\Theta} = \overset{\alpha0\beta}{\Theta} = \overset{\alpha\beta0}{\Theta} = \overset{\alpha\otimes\beta}{\mathbf{1}} \tag{16}$$

and other similar identities for Θ^{-1}. Moreover we have the quasi-inverse properties:

$$\overset{\alpha\beta\gamma}{\Theta}\ \overset{\alpha\beta\gamma}{\Theta^{-1}} = \overset{(\alpha\otimes\beta)\otimes\gamma}{\mathbf{1}} \text{ and } \overset{\alpha\beta\gamma}{\Theta^{-1}}\ \overset{\alpha\beta\gamma}{\Theta} = \overset{\alpha\otimes(\beta\otimes\gamma)}{\mathbf{1}} \tag{17}$$

recalling that, here, $\overset{\alpha\otimes\beta}{\mathbf{1}}$ is simply a projector. Let us now define intertwiners between $\alpha\otimes\beta$ and $\beta\otimes\alpha$ using our basic objects ψ,ϕ:

$$P_{12}\ \overset{\alpha\beta}{R} = \sum_{\gamma\in Phys(\mathcal{A}),m} \lambda_{\alpha\beta\gamma}^{\frac{1}{2}}\ \phi^{\beta\alpha}_{\gamma,m}\psi^{\gamma,m}_{\alpha\beta} \tag{18}$$

$$P_{12}\ \overset{\alpha\beta}{R'^{-1}} = \sum_{\gamma\in Phys(\mathcal{A}),m} \lambda_{\alpha\beta\gamma}^{-\frac{1}{2}}\ \phi^{\beta\alpha}_{\gamma,m}\psi^{\gamma,m}_{\alpha\beta} \tag{19}$$

$$\tag{20}$$

where $R' = \sigma(R)$ and $\lambda_{\alpha\beta\gamma} = (\frac{v_\alpha v_\beta}{v_\gamma})$ where v_α is the Drinfeld casimir, equal to $q^{C^{(2)}_\alpha}$, where $C^{(2)}_\alpha$ is the quadratic Casimir. We will denote in the following $\overset{\alpha\beta}{R} = \sum_i \overset{\alpha}{a}_i\otimes\overset{\beta}{b}_i$ and $\overset{\alpha\beta}{R^{-1}} = \sum_i \overset{\alpha}{c}_i\otimes\overset{\beta}{d}_i$ and use sometimes the notation $R^{(+)} = R$ and $R^{(-)} = R'^{-1}$. Using the hexagonal identities on the $6-j$ symbols it can be shown that:

$$\overset{(\alpha\otimes\beta)\gamma}{R} = \overset{\gamma\alpha\beta}{\Theta}\ \overset{\alpha\gamma}{R}\ \overset{\alpha\gamma\beta}{\Theta^{-1}}\ \overset{\beta\gamma}{R}\ \overset{\alpha\beta\gamma}{\Theta} \tag{21}$$

$$\overset{\alpha(\beta\otimes\gamma)}{R} = \overset{\beta\gamma\alpha}{\Theta^{-1}}\ \overset{\alpha\gamma}{R}\ \overset{\beta\alpha\gamma}{\Theta}\ \overset{\alpha\beta}{R}\ \overset{\alpha\beta\gamma}{\Theta^{-1}} \tag{22}$$

which is simply the analog of the quasitriangularity property of $R-$matrices. This matrix is no more inversible but we have:

$$\overset{\alpha\beta}{R}\ \overset{\alpha\beta}{R^{-1}} = \sigma(\overset{\alpha\otimes\beta}{\mathbf{1}}) \text{ and } \overset{\alpha\beta}{R^{-1}}\ \overset{\alpha\beta}{R} = \overset{\alpha\otimes\beta}{\mathbf{1}}. \tag{23}$$

Let us now study the properties of the antipodal map and develop the analog of the ribbon properties [6]. We will denote by $\overset{\alpha}{A}$ and $\overset{\alpha}{B}$ the matrices defined by $\psi^0_{\bar\alpha\alpha} =< .,\overset{\alpha}{A}. >$ and $\phi^{\alpha\bar\alpha}_0 = (\lambda\to\lambda\sum_i \overset{\alpha}{B}\overset{\alpha}{e}_i\otimes\overset{\bar\alpha}{e^i})$ where $\overset{\alpha}{e}_i$ (resp. $\overset{\bar\alpha}{e^i}$) is a basis of the representation space of α (resp. $\bar\alpha$) and $< .,. >$ is the duality bracket. To choose the normalisation of ϕ and ψs we will impose the ambiant isotopy conditions:

$$\sum_i \theta^{(1)}_i BS(\theta^{(2)}_i)A\theta^{(3)}_i = 1 \text{ and } \sum_i S({\theta^{-1}}^{(1)}_i)A{\theta^{-1}}^{(2)}_i BS({\theta^{-1}}^{(3)}_i) = 1 \tag{24}$$

In order to generalize the known properties relative to the antipode, we will also introduce some notations which will be useful in the following:

$$
\begin{aligned}
\overset{\alpha\beta}{G} &= \sum_{i,j} (S(\overset{\alpha}{\theta_i^{-1(12)}}) \otimes S(\overset{\beta}{\theta_i^{-1(11)}}))(S(\overset{\alpha}{\theta_j^{(2)}}) \otimes S(\overset{\beta}{\theta_j^{(1)}}))(\overset{\alpha}{A} \otimes \overset{\beta}{A})(\overset{\alpha}{\theta_j^{(3)}} \otimes \overset{\beta}{\mathbf{1}})(\overset{\alpha}{\theta_i^{-1(2)}} \otimes \overset{\beta}{\theta_i^{-1(3)}}) \\
\overset{\alpha\beta}{D} &= \sum_{i,j} (\overset{\alpha\otimes\beta}{\theta_i^{(1)}})(\overset{\alpha}{\theta_j^{-1(1)}} \otimes \overset{\beta}{\theta_j^{-1(2)}})(\overset{\alpha}{B} \otimes \overset{\beta}{B})(\overset{\alpha}{\mathbf{1}} \otimes S(\overset{\beta}{\theta_j^{-1(3)}}))(S(\overset{\alpha}{\theta_i^{(3)}}) \otimes S(\overset{\beta}{\theta_i^{(2)}})) \\
\overset{\alpha\beta}{f} &= \sum_{i} (S(\overset{\alpha}{\theta_i^{-1(12)}}) \otimes S(\overset{\beta}{\theta_i^{-1(11)}}))\overset{\alpha\beta}{G}\ (\overset{\alpha\otimes\beta}{\theta_i^{-1(2)} B S(\theta_i^{-1(3)})}) \qquad (25)
\end{aligned}
$$

it can be shown that the latter matrices verify:

$$\overset{\alpha\beta}{f}{}^{-1}\ \overset{\alpha\beta}{G} = \overset{\alpha\otimes\beta}{A} \text{ and } \overset{\alpha\beta}{D}\ \overset{\alpha\beta}{f} = \overset{\alpha\otimes\beta}{B} \qquad (26)$$

$$\phi_\gamma^{\alpha\beta} = \overset{\alpha\beta}{f}{}^{-1}\ {}^t\psi^{\bar\gamma}_{\bar\beta\bar\alpha} \text{ and } \psi^\gamma_{\alpha\beta} = {}^t\phi^{\bar\beta\bar\alpha}_{\bar\gamma}\ \overset{\alpha\beta}{f} \qquad (27)$$

We endly introduce the element u associated to the square of the antipode, defined by:

$$u = \sum_{i,j} S(\theta_i^{-1(2)} B S(\theta_i^{-1(3)})) S(b_j) A a_j \theta_i^{-1(1)}, \qquad (28)$$

u is invertible and

$$
\begin{aligned}
1 &= u \sum_{i,j} S^{-1}(\theta_i^{-1(1)}) S^{-1}(A d_j) c_j \theta_i^{-1(2)} B \theta_i^{-1(3)} = \qquad (29) \\
&= S^2(\sum_{i,j} S^{-1}(\theta_i^{-1(1)}) S^{-1}(A d_j) c_j \theta_i^{-1(2)} B \theta_i^{-1(3)}) u \qquad (30)
\end{aligned}
$$

moreover we have as usual the essential property

$$\forall \xi \in \mathcal{A}, S^2(\xi) = u \xi u^{-1} \qquad (31)$$

and the usual corollaries

$$
\begin{aligned}
& S^2(u) = u \qquad (32) \\
& uS(u) = S(u)u \text{ is central} \qquad (33) \\
& \sum_i S(b_i) A a_i = S(A) u = S(u) u \sum_i S(c_i) A d_i \qquad (34) \\
& \epsilon(u) = 1 \qquad (35)
\end{aligned}
$$

We will denote by v the element satisfying:

$$
\begin{aligned}
& v^2 = uS(u) \qquad (36) \\
& S(v) = v \text{ and } \epsilon(v) = 1. \qquad (37)
\end{aligned}
$$

and by μ the element uv^{-1}. Then it can be shown that:

$$\overset{\alpha\otimes\beta}{\mu} = \overset{\alpha\beta}{f}{}^{-1}(S \otimes \overset{\alpha\beta}{S})(\sigma(f)) \;\; (\overset{\alpha}{\mu} \otimes \overset{\beta}{\mu}) \tag{38}$$

using the latter notations it can be shown that $\phi_0^{\alpha\bar{\alpha}} =< S(A)\mu \; . \; , \; . >$ and $\psi^0_{\bar{\alpha}\alpha} = (\lambda \to \lambda \sum_i \overset{\bar{\alpha}}{e^i} \otimes \mu^{-1} S(B) \overset{\alpha}{e_i})$. In the following the q-dimension of the representation α will be defined by $[d_\alpha] = tr_\alpha(S(A)\mu B)$.

Now, using the latter framework we can give a well defined construction of the quantum group in the dual version for any value of q. As a vector space this algebra, called Γ, is generated by $\{\overset{\alpha}{g^i_j}$ with $\alpha \in Phys(\mathcal{A})$ and $i, j = 1 \cdots dim(\alpha)\}$ and the product is simply defined by:

$$\overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \sum_{\gamma \in Phys(\mathcal{A})} \phi^{\alpha\beta}_\gamma \;\; \overset{\gamma}{g} \;\; \psi^\gamma_{\alpha\beta}. \tag{39}$$

This product is not associative but verify:

$$\overset{\alpha\beta\gamma}{\Theta}{}_{123}((\overset{\alpha}{g}_1 \overset{\beta}{g}_2)\overset{\gamma}{g}_3) = (\overset{\alpha}{g}_1(\overset{\beta}{g}_2 \overset{\gamma}{g}_3)) \overset{\alpha\beta\gamma}{\Theta}{}_{123}. \tag{40}$$

Moreover we have the exchange relation:

$$\overset{\alpha\beta}{R}_{12} \;\; \overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \overset{\beta}{g}_2 \overset{\alpha}{g}_1 \;\; \overset{\alpha\beta}{R}_{12}. \tag{41}$$

This algebra can be equiped with a coproduct and a counity:

$$\Delta(\overset{\alpha}{g^i_j}) = \sum_i \overset{\alpha}{g^i_k} \;\otimes\; \overset{\alpha}{g^k_j} \text{ and } \epsilon(\overset{\alpha}{g^i_j}) = \delta^i_j. \tag{42}$$

Moreover it can be shown that the antipodal map S defined to be the linear map verifying $S(\overset{\alpha}{g^i_j}) = \overset{\bar{\alpha}}{g^j_i}$ owns the properties:

$$S(g)^i_k A^k_l g^l_j = A^i_j \tag{43}$$

$$(S(A)\mu)^k_l g^l_j S(g)^i_k = (S(A)\mu)^i_j \tag{44}$$

$$S(\overset{\beta}{g}_2) S(\overset{\alpha}{g}_1) = \overset{\alpha\beta}{f}{}_{12} \; S(\overset{\alpha}{g}_1 \overset{\beta}{g}_2) \; \overset{\alpha\beta}{f}{}^{-1}{}_{12} \tag{45}$$

$$S^2(\overset{\alpha}{g}) = \overset{\alpha}{\mu}\overset{\alpha}{g}\overset{\alpha}{\mu}{}^{-1} \tag{46}$$

Our aim is now to define as in our first paper the gauge theory associated to this gauge symmetry algebra.

Let Σ be a compact connected oriented surface with boundary $\partial\Sigma$ and let $\mathcal{T}$ be a triangulation of Σ. Let us denote by $\mathcal{F}$ the oriented faces of $\mathcal{T}$, by $\mathcal{L}$ the

set of edges counted with their orientation. If l is an interior link, $-l$ will denote the opposite link. We have $\mathcal{L} = \mathcal{L}^{int} \cup \mathcal{L}^{\partial\Sigma}$, where $\mathcal{L}^{int}, \mathcal{L}^{\partial\Sigma}$ are respectively the set of interior edges and boundary edges.

Finally let us also define $\mathcal{V}$ to be the set of points (vertices) of this triangulation, $\mathcal{V} = \mathcal{V}^{int} \cup \mathcal{V}^{\partial\Sigma}$, where $\mathcal{V}^{int}, \mathcal{V}^{\partial\Sigma}$ are respectively the set of interior vertices and boundary vertices.

If l is an oriented link it will be convenient to write $l = xy$ where y is the departure point of l and x the end point of l. We will write $y = d(l)$ and $x = e(l)$.

Definition 1 (gauge symmetry algebra) *Let us define for $z \in \mathcal{V}$, the Hopf algebra $\Gamma_z = \Gamma \times \{z\}$ and $\hat{\Gamma} = \bigotimes_{z\in\mathcal{V}} \Gamma_z$. This Hopf algebra was called in [2] "the gauge symmetry algebra."*

If x is a vertex we shall write $\overset{\alpha}{g}_x$ to denote the embedding of the element $\overset{\alpha}{g}$ in Γ_x.

In order to define the non commutative analogue of algebra of gauge fields we have to endow the triangulation with an additional structure[1], an order between links incident to a same vertex, the *cilium order*.

Definition 2 (Ciliation) *A ciliation of the triangulation is an assignment of a cilium c_z to each vertex z which consists in a non zero tangent vector at z. The orientation of the Riemann surface defines a canonical cyclic order of the links admitting z as departure or end point. Let l_1, l_2 be links incident to a common vertex z, the strict partial cilium order $<_c$ is defined by:*

$l_1 <_c l_2$ if $l_1 \neq l_2, -l_2$ and the unoriented links c_z, l_1, l_2 appear in the cyclic order defined by the orientation.

If l_1, l_2 are incident to a same vertex z we define:

$$\epsilon(l_1, l_2) = \begin{cases} +1 & \text{if } l_1 <_c l_2 \\ -1 & \text{if } l_2 <_c l_1 \end{cases}$$

Definition 3 (Gauge fields algebra) *The algebra of gauge fields [4][2] Λ is the non associative algebra generated by the formal variables $\overset{\alpha}{u}(l)^i_j$ with $l \in \mathcal{L}, \alpha \in Phys(\mathcal{A}), i, j = 1 \cdots dim(\alpha)$ and satisfying the following relations:*

Commutation rules

$$\overset{\alpha\beta}{R}_{12} \overset{\alpha}{u}(yx)_1 \overset{\beta}{u}(yz)_2 = \overset{\beta}{u}(yz)_2 \overset{\alpha}{u}(yx)_1 \tag{47}$$

$$\overset{\alpha}{u}(xy)_1 (S \otimes id)(\overset{\alpha\beta}{R}_{12}) \overset{\beta}{u}(yz)_2 = \overset{\beta}{u}(yz)_2 \overset{\alpha}{u}(xy)_1 \tag{48}$$

$$\overset{\alpha}{u}(xy)_1 \overset{\beta}{u}(zy)_2 (S \otimes S)(\overset{\alpha\beta}{R}_{12}) = \overset{\beta}{u}(zy)_2 \overset{\alpha}{u}(xy)_1 \tag{49}$$

$$\forall\ (yx), (yz) \in \mathcal{L}\, x \neq z \text{ and } xy <_c yz$$

$$\overset{\alpha}{u}(l) \overset{\alpha}{A} \overset{\alpha}{u}(-l) = \overset{\alpha}{B} \tag{50}$$

$$
\begin{aligned}
&\forall\ l \in \mathcal{L}^i, \\
&\overset{\alpha}{u}(xy)_1 \overset{\beta}{u}(zt)_2 = \overset{\beta}{u}(zt)_2 \overset{\alpha}{u}(xy)_1 \\
&\forall\ x, y, z, t \textit{ pairwise distinct in } \mathcal{V}
\end{aligned} \tag{51}
$$

Decomposition rule

$$
\overset{\alpha}{u}(l)_1 \overset{\beta}{u}(l)_2 = \sum_{\gamma,m} \phi^{\alpha,\beta}_{\gamma,m} \overset{\gamma}{u}(l) \psi^{\gamma,m}_{\beta,\alpha} \overset{\alpha\beta}{f^{-1}} P_{12}, \tag{52}
$$

$$
\overset{0}{u}(l) = 1,\ \forall l \in \mathcal{L}. \tag{53}
$$

Quasi-associativity *Let $\mathcal{M}_P$ be a monomial of gauge fields algebra elements with a certain parenthesing P. For each vertex x of the triangulation we construct a tensor product of representations of $\mathcal{A}$ by replacing each $\overset{\alpha}{u_l}$ in the monomial by the vector space α (resp. $\bar{\alpha}$ resp. 0) depending on whether x is the endpoint (resp. departure point resp. not element) of the edge l, and keeping the previous parenthesing. Let us consider two different parenthesing P_1 and P_2 of the same monomial. We can construct for both, as described before, the corresponding vector spaces for each x and deduce the intertwiner Θ_x relating them. The relation of quasi-associativity is then simply:*

$$
(\prod_{x\in\mathcal{V}} \Theta_x)\mathcal{M}_{P_1} = \mathcal{M}_{P_2} \tag{54}
$$

Proposition 1 (Gauge covariance) *Λ is a right $\hat{\Gamma}$ comodule defined by the morphism of algebra $\Omega : \Lambda \to \Lambda \otimes \hat{\Gamma}$:*

$$
\Omega(\overset{\alpha}{u}(xy)) = \overset{\alpha}{g}_x \overset{\alpha}{u}(xy) S(\overset{\alpha}{g}_y). \tag{55}
$$

The definition relations of the gauge fields algebra are compatible to the coaction of the gauge symmetry algebra.

The subalgebra of gauge invariant elements of Λ is denoted Λ^{inv}.

2.2 Invariant measure, holonomies, zero-curvature projector

It was shown (provided some assumption on the existence of a basis of Λ of a special type) [4][2] that there exists a unique non zero linear form $h \in \Lambda^\star$ satisfying:

1. (invariance) $(h \otimes id)\Omega(A) = h(A) \otimes 1\ \forall A \in \Lambda$
2. (factorisation) $h((A)(B)) = h(A)h(B)$
   $\forall A \in \Lambda_X, \forall B \in \Lambda_Y, \forall X, Y \subset L,\ (X \cup -X) \cap (Y \cup -Y) = \emptyset$

(we have used the notation Λ_X for $X \subset \mathcal{L}$ to denote the subalgebra of Λ generated as an algebra by $\overset{\alpha}{u}_l$ with $l \in X$).

It can be evaluated on any element using the formula:

$$h(\overset{\alpha}{u}(x,y)^i_j) = \delta_{\alpha,0} \tag{56}$$

where 0 denotes the trivial representation of dimension 1, i.e 0 is the counit.

It was convenient to use the notation $\int dh$ instead of h. We obtained the important formula:

$$h(\overset{\alpha}{u}(y,x)_1 (S \otimes id)(\overset{\alpha\alpha}{R}_{12}) v_\alpha^{-1} \overset{\alpha}{u}(x,y)_2) = \frac{1}{[d_\alpha]} P_{12} \overset{\alpha}{B}_1. \tag{57}$$

A path P (resp. a loop P) is a path (resp.a loop) in the graph attached to the triangulation of Σ, given by the collection of its vertices, it will also denote equivalently the continuous curve (resp. loop) in Σ defined by the links of P. In this article we will denote by $P = [x_n, x_{n-1}, \cdots, x_1, x_0]$ a link with departure point x_0 and end point x_n. Following the definition for links, the departure point of P is denoted $d(P)$ and its endpoint $e(P)$. The set of vertices (resp. edges) of the path P is denoted by $\mathcal{V}(P)$ (resp. $\mathcal{L}(P)$), the cardinal of this set is called the "length" of P and will be denoted by $Length(P)$.

Properties of path and loops such as self intersections, transverse intersections will always be understood as properties satisfied by the corresponding curves on Σ.

Let $P = [x_n, ..., x_0]$ be a path, we defined the sign $\epsilon(x_i, P)$ to be -1 (resp. 1) if $x_{i-1}x_i <_c x_i x_{i+1}$ (resp. $x_i x_{i+1} <_c x_{i-1} x_i$).

Definition 4 (Holonomies and Wilson loops) *If P is a simple path $P = [x_n, \cdots, x_0]$ with $x_0 \neq x_n$, we define the holonomy along P by*

$$\overset{\alpha}{u}_P = v_\alpha^{\frac{1}{2}\sum_{i=1}^{n-1} \epsilon(x_i,P)} (\overset{\alpha}{u}(x_n x_{n-1}) \overset{\alpha}{A} \overset{\alpha}{u}(x_{n-1} x_{n-2}) \overset{\alpha}{A} \cdots \overset{\alpha}{A} \overset{\alpha}{u}(x_1 x_0)). \tag{58}$$

When C is a simple loop $C = [x_{n+1} = x_0, x_n, \cdots, x_0]$, we define the holonomy along C by

$$\overset{\alpha}{u}_C = v_\alpha^{\frac{1}{2}(\sum_{i=1}^{n} \epsilon(x_i,C) - \epsilon(x_0,C))} (\overset{\alpha}{u}(x_0 x_n) \overset{\alpha}{A} \overset{\alpha}{u}(x_n x_{n-1}) \overset{\alpha}{A} \cdots \overset{\alpha}{A} \overset{\alpha}{u}(x_1 x_0)). \tag{59}$$

We define an element of Λ, called Wilson loop attached to C :

$$\overset{\alpha}{W}_C = tr_\alpha(S(\overset{\alpha}{A}) \overset{\alpha}{\mu} \overset{\alpha}{u}_C). \tag{60}$$

the interior parenthesing being irrelevant because the loop is simple (the vector space attached to a vertex occurs, first with α, second with $\bar{\alpha}$ and the other times in the trivial representation) and moreover we have the relations (16). We will also use the notation $\overset{\alpha}{W}_C = \overset{\alpha}{W}_{[x_0, x_n \cdots, x_1]}$.

The properties shown in our first paper are easily generalized:

Proposition 2 (Properties of Wilson loops) *The element* $\overset{\alpha}{W}_C$ *is gauge invariant and moreover it does not depend on the departure point of the loop* C*. Moreover it verifies the fusion equation:*

$$\overset{\alpha}{W}_C \overset{\beta}{W}_C = \sum_{\gamma \in Phys(A)} N^{\gamma}_{\alpha\beta} \overset{\gamma}{W}_C \tag{61}$$

Proof: The gauge invariance is quite obvious because of relations (43), (44). To show the cyclicity property we must put our Wilson loop in another form called "expanded form" in our first paper. Using relations (34),(23) we easily obtain:

$$\overset{\alpha}{W}(C) = v_\alpha^{-\frac{1}{2}(\sum_{x\in C}\epsilon(x,C))} tr_{\alpha^{\otimes n}}((S(\overset{\alpha}{A}) \overset{\alpha}{\mu})^{\otimes n} \prod_{i=n}^{1} P_{ii-1} \times \tag{62}$$
$$\times (\prod_{i=n}^{1} \overset{\alpha}{u}(x_{j+1}x_j)_j (S\otimes id)(R_{jj-1}^{(\epsilon(x_j,C))})) \overset{\alpha}{u}(x_1x_0)_1).$$

In this form the cyclicity invariance is obvious using the commutation relations (48).
The fusion relation is less trivial to show. We first show a lemma describing the decomposition rules for holonomies:

$$(\overset{\alpha}{u}_P)_1(\overset{\beta}{u}_P)_2 = \phi^{\alpha\beta}_{\gamma} \; \overset{\gamma}{u}_P \; \psi^{\gamma}_{\beta\alpha} P_{12} \overset{\alpha\beta}{f^{-1}}_{21}. \tag{63}$$

Indeed, using (54)(48), we easily obtain for a path $P=[x,y,z]$:

$$\begin{aligned}
&(v_\alpha^{\frac{1}{2}\epsilon(y,P)}(\overset{\alpha}{u}(xy)\overset{\alpha}{A}\overset{\alpha}{u}(yz))_1 (v_\beta^{\frac{1}{2}\epsilon(y,P)}(\overset{\beta}{u}(xy)\overset{\beta}{A}\overset{\beta}{u}(yz))_2 = \\
&= \sum_{i,j,k,l} v_\alpha^{\frac{1}{2}\epsilon(y,P)} v_\beta^{\frac{1}{2}\epsilon(y,P)} (\overset{\alpha}{u}(xy)_1 \overset{\beta}{u}(xy)_2)(S(\theta_l^{(1)})A\theta_i^{(1)} b_j \theta_k^{-1(2)}\theta_l^{(31)})_1 \times \\
&\times (S(\theta_l^{(2)})S(\theta_k^{-1(1)})S(a_j)S(\theta_i^{(2)})A\theta_i^{(3)}\theta_k^{-1(3)}\theta_l^{(32)})_2 (\overset{\alpha}{u}(yz)_1 \overset{\beta}{u}(yz)_2) = \\
&= v_\alpha^{\frac{1}{2}\epsilon(y,P)} v_\beta^{\frac{1}{2}\epsilon(y,P)} (\overset{\alpha}{u}(xy)_1 \overset{\beta}{u}(xy)_2) \overset{\alpha\beta}{G}_{21} \overset{\alpha\beta}{R'^{-1}} (\overset{\alpha}{u}(yz)_1 \overset{\beta}{u}(yz)_2)
\end{aligned}$$

the last equality is obtained by using successively (22) and (15). Now, using (26)(18), we obtain the announced result for a two links path. Proceeding by induction we can prove it for any simple open path.

Let us now consider the loop C as formed by two pieces $[xy]$ and $[yx]$, we have, using the same properties as before:

$$
\begin{aligned}
&(v_\alpha^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))} tr_\alpha(S(\overset{\alpha}{A})\, \overset{\alpha}{\mu}\overset{\alpha}{u}\,(xy)\overset{\alpha}{A}\,\overset{\alpha}{u}\,(yx)))(v_\beta^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))} tr_\beta(S(\overset{\beta}{A})\, \overset{\beta}{\mu}\overset{\beta}{u}\,(xy)\overset{\beta}{A}\,\overset{\beta}{u}\,(yx))) = \\
&= \sum_{\substack{p,m,l,i\\ q,n,j,k}} (v_\alpha v_\beta)^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))} tr_{\alpha\beta}((S(\theta_p^{-1(2)})S(\theta_m^{(3)})S(c_l)S(\theta_i^{-1(2)})S(A)\mu\theta_i^{-1(1)}\theta_m^{(1)}\theta_p^{-1(11)})_1 \times \\
&\times(S(\theta_p^{-1(3)})S(A)\mu\theta_i^{-1(3)}d_l\theta_m^{(2)}\theta_p^{-1(12)})_2(\overset{\alpha}{u}\,(xy)_1\,\overset{\beta}{u}\,(xy)_2)(S(\theta_q^{-1(11)})S(\theta_n^{(1)})S(\theta_j^{-1(1)}) \times \\
&\times A\theta_j^{-1(2)}b_k\theta_n^{(3)}\theta_q^{-1(2)})_1(S(\theta_q^{-1(12)})S(\theta_n^{(2)})S(a_k)S(\theta_j^{-1(3)})A\theta_q^{-1(3)})_2(\overset{\alpha}{u}\,(yx)_1\,\overset{\beta}{u}\,(yx)_2)) = \\
&= (v_\alpha v_\beta)^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))} tr_{\alpha\beta}((S\otimes S)(G_{21}R_{12})(\mu\otimes\mu)(\overset{\alpha}{u}\,(xy)_1\,\overset{\beta}{u}\,(xy)_2)(G_{21}R_{21}^{-1}) \times \\
&\times(\overset{\alpha}{u}\,(yx)_1\,\overset{\beta}{u}\,(yx)_2)) = \\
&= \sum_\gamma N_\gamma^{\alpha\beta} v_\gamma^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))} tr_\gamma(S(\overset{\gamma}{A})\,\overset{\gamma}{\mu}\,\overset{\gamma}{u}(xy)\overset{\gamma}{A}\overset{\gamma}{u}(yx))
\end{aligned}
$$

which ends the proof of the theorem. □

Proposition 3 (commutation properties) *It can also be shown that* $[\overset{\alpha}{W}_C\,, \overset{\beta}{W}_{C'}] = 0$ *for all simple loops* C, C' *without transverse intersections.*

Although the structure of the algebra Λ depends on the ciliation, it has been shown in [4] that the algebra Λ^{inv} does not depend on it up to isomorphism. This is completely consistent with the approach of V.V.Fock and A.A.Rosly: in their work the graph needs to be endowed with a structure of ciliated fat graph in order to put on the space of graph connections $\mathcal{A}^l$ a structure of Poisson algebra compatible with the action of the gauge group G^l. However, as a Poisson algebra $\mathcal{A}^l/G^l$ is canonically isomorphic to the space $\mathcal{M}^G$ of flat connections modulo the gauge group, the Poisson structure of the latter being independent of any choice of r-matrix [1].

Definition 5 (zero-curvature projector) *We introduced a Boltzmann weight attached to any simple loop* C *and defined by:*

$$\delta_C = \sum_{\alpha\in Phys(A)} [d_\alpha]\,\overset{\alpha}{W}_C\,. \tag{64}$$

Proposition 4 *This element satisfies the flatness relation :*

$$\delta_C\,\overset{\alpha}{u}_C{}^i_j \quad = \quad \overset{\alpha}{B}{}^i_j\delta_C. \tag{65}$$

moreover we have

$$(\frac{(\delta_C)}{\sum_{\alpha\in Phys(\mathcal{A})}[d_\alpha]^2})^2 \quad = \quad (\frac{(\delta_C)}{\sum_{\alpha\in Phys(\mathcal{A})}[d_\alpha]^2}). \tag{66}$$

Proof: Using the same properties as in the computation of fusion relations, we obtain :

$$\sum_{\alpha\in Phys(\mathcal{A})}[d_\alpha](v_\alpha^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))}tr_\alpha(S(\overset{\alpha}{A})\overset{\alpha\alpha}{\mu u}(xy)\overset{\alpha}{A}\overset{\alpha}{u}(yx)))\times$$
$$\times(v_\beta^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))}tr_\beta(S(\overset{\beta}{A})\overset{\beta\beta}{\mu u}(xy)\overset{\beta}{A}\overset{\beta}{u}(yx)))=$$
$$=\sum_{\substack{\alpha\in Phys(\mathcal{A})\\ p,m,l,i,q,n,j,k}}[d_\alpha](v_\alpha v_\beta)^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))}tr_{\alpha\beta}((S(\theta_p^{-1(2)})S(\theta_m^{(3)})S(c_l)S(\theta_i^{-1(2)})S(A)\mu\theta_i^{-1(1)}\theta_m^{(1)}\times$$
$$\theta_p^{-1(11)})_1(\theta_i^{-1(3)}d_l\theta_m^{(2)}\theta_p^{-1(12)})_2(\overset{\alpha}{u}(xy)_1\overset{\beta}{u}(xy)_2)(S(\theta_q^{-1(11)})S(\theta_n^{(1)})S(\theta_j^{-1(1)})A\theta_j^{-1(2)}b_k\theta_n^{(3)}\theta_q^{-1(2)})_1\times$$
$$\times(S(\theta_q^{-1(12)})S(\theta_n^{(2)})S(a_k)S(\theta_j^{-1(3)})A\theta_q^{-1(3)})_2(\overset{\alpha}{u}(yx)_1\overset{\beta}{u}(yx)_2)S(\theta_p^{-1(3)})_2)=$$
$$=\sum_{\alpha,\gamma\in Phys(\mathcal{A})}[d_\alpha]\sum_{p,m}tr_{\alpha\beta}((S(\theta_m^{(3)})S(A)\mu\theta_m^{(2)}\theta_p^{-1(12)})_1\times$$
$$\times(\theta_m^{(1)}\theta_p^{-1(11)})_2\phi_\gamma^{\beta\alpha}\ \overset{\gamma}{u}(xy)A\overset{\gamma}{u}(yx)v_\gamma^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))}\ \psi_{\beta\alpha}^\gamma f_{21}^{-1}\ S(\theta_p^{-1(2)})_1S(\theta_p^{-1(3)})_2)$$

the last equality uses again the quasitriangularity properties. Then we obtain, for any matrix V in $End(\alpha)$:

$$\delta_C\ tr_\alpha(S(A)\overset{\alpha}{\mu}V\overset{\alpha}{u}_C)=$$
$$=\sum_{\gamma\in Phys(\mathcal{A})}\sum_{p,m}tr_\gamma(\ (\sum_{\alpha\in Phys(\mathcal{A})}[d_\alpha]\psi_{\beta\alpha}^\gamma f_{21}^{-1}(S(\theta_p^{-1(2)})S(\theta_m^{(3)})S(A)\mu\theta_m^{(2)}\theta_p^{-1(12)})_1\times$$
$$\times(S(\theta_p^{-1(3)})S(A)\mu\ V\ \theta_m^{(1)}\theta_p^{-1(11)})_2\phi_\gamma^{\beta\alpha})\ (\overset{\gamma}{u}(xy)A\overset{\gamma}{u}(yx)v_\gamma^{\frac{1}{2}(\epsilon(y,P)-\epsilon(x,C))})\ )$$

Due to the intertwining properties and the normalizations mentioned before of the ϕ,ψs we can conclude (see [4]) that it does exist some complex coefficients $A(\alpha\beta\gamma)$ such that:

$$id_K\psi_{\beta\bar\beta}^0=\sum_\alpha A(\alpha\beta\gamma)\psi_{\beta\alpha}^\gamma\psi_{\bar\beta\gamma}^\alpha \tag{67}$$

and

$$\phi_{\beta\alpha}^\gamma=A(\alpha\beta\gamma)\frac{[d_\gamma]}{[d_\alpha]}(\psi_{\bar\beta\gamma}^\alpha\overset{\beta\bar\beta\gamma}{\Theta}\otimes id_\beta)(\phi_0^{\beta\bar\beta}\otimes id_\gamma) \tag{68}$$

we obtain

$$\sum_{p,m}\sum_{\alpha\in Phys(\mathcal{A})}[d_\alpha]\psi_{\beta\alpha}^\gamma f_{21}^{-1}(S(\theta_p^{-1(2)})S(\theta_m^{(3)})S(A)\mu\theta_m^{(2)}\theta_p^{-1(12)})_1\times$$
$$\times(S(\theta_p^{-1(3)})S(A)\mu\ V\ \theta_m^{(1)}\theta_p^{-1(11)})_2\phi_\gamma^{\beta\alpha}=[d_\gamma]id_\gamma tr_\beta(S(A)\mu\ V\ B)$$

then for any matrix V we have $\delta_C\ tr_\alpha(\overset{\alpha}{S}(A)\ \overset{\alpha}{\mu}\ \overset{\alpha}{V}\ \overset{\alpha}{u}_C) = \delta_C\ tr_\alpha(\overset{\alpha}{S}(A)\ \overset{\alpha}{\mu}\ \overset{\alpha}{V}\overset{\alpha}{B})$, and the linear independance of the generators of our algebra ensures the final result.

The last formula of the proposition is a trivial consequence of the last result. $\square$

We were led to define an element that we called $a_{YM} = \prod_{f\in\mathcal{F}} \delta_{\partial f}$. This element is the non commutative analogue of the projector on the space of flat connections.

In [4][2] it was proved that $\delta_{\partial f}$ is a central element of Λ^{inv} and the algebra $\Lambda_{CS} = \Lambda^{inv} a_{YM}$ was shown to be independant, up to isomorphism, of the triangulation. The proof is based on the lemma of decomposition rules of the holonomies shown before and on the quite obvious property: let C_1 and C_2 be two simple contractile loops which interiors are disjoint and with a segment $[xy]$ of their boundary in common:

$$\int dh(u(xy))\ \overset{\alpha}{W}_{C_1} \overset{\beta}{W}_{C_2} = \delta_{\alpha,\beta}\ \overset{\alpha}{W}(C_1 \# C_2) \tag{69}$$

As a result it was advocated that Λ_{CS} is the algebra of observables of the Chern Simons theory on the manifold $\Sigma\times[0,1]$. This is supported by the topological invariance of Λ_{CS} (i.e this algebra depends only on the topological structure of the surface Σ) and the flatness of the connection.

Our aim is now to construct in the algebra Λ_{CS} the observables associated to any link in $\Sigma \times [0,1]$.

2.3 Links, chord diagrams and quantum observables

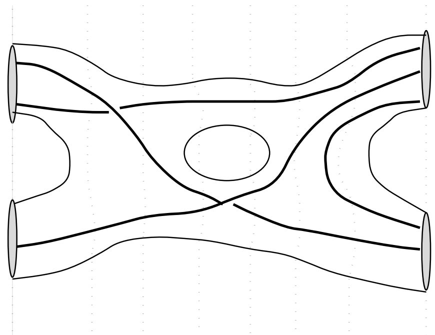
In the following subsection and in the chapter 3 the computations will be made in the case of q generic to simplify the notations but the generalization to q root of unity can be made exactly in the same way.

We will consider a compact connected surface Σ with boundary $\partial\Sigma$. The boundary is a set of disjoint simple closed curves which are designed to be "In" or "Out". Let us draw some oriented curves on the surface Σ defining a link L , assuming that their boundary is contained in $\partial\Sigma$ and with simple, transverse intersections, with the specification of over- or undercrossing at each intersection. We will also consider that representations of the quantum group are attached to connected components of the link.

The data (surface with boundary + colored link) will be called "striped surface", the data of "In" (resp. "Out") boundary of Σ and L with corresponding colors will be called the "In state" (resp. the "Out state") of the striped surface.

To describe such objects we will choose a Morse function which gives a time direction and the set of "equitime planes" $(\mathcal{P}_t)_{t_i\leq t\leq t_f}$ cutting the surface. An equitime plane P_t divides the surface in two parts called respectively "future"

and "past". On any simple curve drawn on an equitime plane the time direction give us an orientation of the curve, if we impose moreover a departure point x on this curve we are able to decide if a point z is on the left (resp. on the right) of another point z', if x, z, z' (resp. x, z', z) appear in the order given by this orientation. We will consider the surface in a canonical position defined by the following conditions. The intersection between the surface and the plane for $t < t_i$ or $t > t_f$ is empty. The "In" (resp. "out") boundary is contained in the $t = t_i$ (resp. $t = t_f$) plane. The intersection between the $(\mathcal{P}_t)_{t_i \leq t \leq t_f}$ and the surface is a set of disjoint simple closed curves $(C_i^t)_{i=1,\ldots,n(t)}$ (not necessary disjoint at the singular times), where $n(t)$ is the number of connected components of $\Sigma \cap \mathcal{P}_t$. We will call "$\varphi^3-$diagram" of the surface a graph drawn on it which intersections with $(\mathcal{P}_t)_{t_i \leq t \leq t_f}$ determine departure points on each closed curve in these sets. We impose that the φ^3-diagram never turn around any handle of the surface.
Our aim is now to define a ciliated fat graph which will encode the topology of the striped surface,i.e. this decomposition involves only contractile plaquettes, it is sufficiently fine to allow us to put the link on the graph in a generic position and allow us to distinguish two situations related by a Dehn twist of the surface. We then decompose the surface in blocks, their number being chosen with respect to the singularities of the Morse function (considered as a function over the points of the surface and of the link). The information contained in the Morse function is not sufficient to deal with the problem of possible non trivial cycles of L around handles of Σ. We will rule out this problem by adding fictively two disjoint φ^3-diagrams of Σ to the link L, the intersections between the link and the φ^3-diagrams will detect the rotation of the link around an handle of Σ, we will then refine the decomposition with respect to these datas. We will assume that the singularities of the Morse function f, considered now as a function of the points of the surface, points of the link and points of the φ^3-diagrams, correspond to different times. We will denote by $t_0 = t_i, t_1, \cdots, t_{n-1}, t_n = t_f$ the different instants corresponding to the singularities of the Morse function. We will consider a decomposition of the surface and of the link in "elementary blocks" $\mathcal{B}_0, \cdots, \mathcal{B}_n$ corresponding to the subdivision $[t_i, t_f] = [\tau_0, \tau_1] \cup [\tau_1, \tau_2] \cup \cdots \cup [\tau_n, \tau_{n+1}]$ where $\tau_0 = t_0, \tau_1 = \frac{1}{2}(t_0 + t_1), \cdots, \tau_n = \frac{1}{2}(t_{n-1} + t_n), \tau_{n+1} = t_n$ are called "cutting times". An example of a striped surface with the block decomposition described before is shown in the following figure:



We will consider a triangulation $\mathcal{T}$ and a ciliation induced by this block decomposition. The set $\mathcal{V}$ of vertices of $\mathcal{T}$ contains all elements of the sets $L \cap \mathcal{P}_{\tau_k}$ and the singularities of the Morse function considered as a function of the link and of the surface. The edges of the triangulation are either a segment of the link or of the φ^3-diagram between two consecutiv vertices, or a segment drawn on the surface at the same time τ_i between two emerging strings. The plaquettes of the triangulation are the connected regions of Σ surrounded by the edges described before. Let us denote by (x_l^k) the intersections of the link with the cutting planes $(\mathcal{P}_{\tau_k})$. The ciliation is chosen to be: at each vertex x_l^k, directed to the past and to the left, just "before" the equitime line, and at each crossing of the oriented link, between the two outgoing strands. We will choose a practical indexation satisfying the following properties: $x_l^k \in C_i^{\tau_k}$ for $l \in \{1 + \sum_{j<i} Length(C_j^{\tau_k}), \cdots \cdots, \sum_{j \leq i} Length(C_j^{\tau_k})\}$ and the x_l^k belonging to the same $C_i^{\tau_k}$ are ordered from the departure point and from left to right. The set of all (x_l^k) for a given k is denoted by $\mathcal{V}_{\tau_k}$. We will say that, at a vertex of L, the orientation of the link is "in the sense of time" or "against the sense of time" according to the position of the link with respect to the cutting plane, the two corresponding subset of $\mathcal{V}$ will be respectively denoted by $\mathcal{V}_-$ and $\mathcal{V}_+$. The set of all edges of $\mathcal{T}$ is again denoted by $\mathcal{L}$ and we will denote by $\mathcal{L}_{\tau_k}$ (resp. $\mathcal{L}_{\tau_k<t<\tau_{k+1}}$) the set of edges contained in $\mathcal{P}_{\tau_k}$ (resp. between $\mathcal{P}_{\tau_k}$ and $\mathcal{P}_{\tau_k+1}$). The elements of $\mathcal{L}_{\tau_k<t<\tau_{k+1}}$ will be currently denoted as l_j^k. We index the elements of this set with the following rule: begining from the departure point fixed by the φ^3-diagram we order the links in an obvious way from left to right if they do not cross themselves and, at a crossing, the first one is that which overcross the other. At last we will denote by $\mathcal{F}_{\tau_k \leq t \leq \tau_{k+1}}$ the set of faces of the triangu lation corresponding to the block $\mathcal{B}_k$. To each element l of $\mathcal{L}$ we will associate a subset of $\mathcal{F}$ called $Present(l)$ defined by the following rule: if l belongs to a crossing and is the overcrossing (resp.undercrossing) strand, $Present(l)$ is the set of the four plaquettes surrounding the crossing (resp. the set is empty), if l is an annihilation or a creation, $Present(l)$ is the set of the two surrounding plaquettes, elsewhere $Present(l)$ contains only the plaquette just at the left of l. We then define $Past(l)$ to be the subset of $\mathcal{F} \setminus Present(l)$ such that $P \in Past(l)$ if it is on the left of l in the same block or anywhere in

a past block. A link in $\Sigma\times[0,1]$ is an embedding of $(S^1)^{\times p'}\times([0,1])^{\times p''}$ into $\Sigma\times[0,1]$, with $\partial L\subset\partial\Sigma\times[0,1]$. On the set of links we can define a composition law, denoted $*$ defined as follows:let L and L' be to links in $\Sigma\times[0,1]$ considered up to ambiant isotopy. We define $L*L'$ to be the link obtained by putting L in $\Sigma\times[\frac{1}{2},1]$ and L' in $\Sigma\times[0,\frac{1}{2}]$. This composition is associative and admit the empty link as unit element. This composition law is commutative if and only if Σ is homeomorphic to the sphere. Let us denote by $(\overset{i}{L})_{i=1\cdots p}$ the connected components of the link L, $\alpha_i\in Phys(\mathcal{A})$ the colour of this component, and by $\overset{i}{P}$ the colored loop obtained by projecting $\overset{i}{L}$ on Σ. It is very convenient to associate to the link L a coloured chord diagram C [12] which will encode intersections of the loops. This chord diagram is constructed as follows: the projection of the link on Σ defines p' colored loops and p'' coloured open paths (with boundary on the boundary of Σ) on Σ with transverse intersections, this configuration of paths defines uniquely a coloured chord diagram by the standard construction. Let us denote by $(\overset{i}{S})_{i=1\cdots p=p'+p''}$ the coloured circles and arcs of the chord diagram corresponding to loops and open paths belonging to the link (we will call abusively "circles" the circles or the arcs of the chord diagrams). The family of coloured circles (resp. arcs) $(\overset{i}{S})_{i=1\cdots p'}$ (resp. $(\overset{i}{S})_{i=p'+1\cdots p}$) will be denoted $\mathcal{C}_1$ (resp. $\mathcal{C}_2$). Each circle $\overset{i}{S}$ is oriented, we will denote by $(\overset{i}{y}_j)_{j=1\cdots n_i}$ the intersection points of the circle $\overset{i}{S}$ with the chords. We will assume that they are labelled with respect to the cyclic order defined by the orientation of the circles. Let $Y=\cup_{i=1}^{p}\{\overset{i}{y}_j,j=1\cdots n_i\}$, we define a relation $\sim$ on the set Y by : $y\sim y'$ if and only if y and y' are connected by a chord. We will denote by φ the immersion of the chord diagram in Σ, in particular we have $\overset{i}{P}=\varphi(\overset{i}{S})$. Every intersection point of the projection of L on Σ have exactly two inverse images by φ in the chord diagram and these points are linked by a unique chord. We will denote by $\overset{i}{z}\overset{k}{_j}\in\overset{i}{S}$ the points such that $\overset{i}{z}\overset{k}{_j}\in]\overset{i}{y}_j\overset{i}{y}_{j-1}[$ and $\varphi(\overset{i}{z}\overset{k}{_j})$ is a vertex of the triangulation corresponding to the cutting time τ_k. We will denote by Z_i the set of all points of type z in the $i-$th component, Z the union of these sets, $Z^{\partial\Sigma}$ the subset of Z formed by the points of Z which belong to the boundary of Σ. Let us denote by $\mathcal{S}_i$ the family of segments forming the corresponding circle and $\mathcal{S}$ the union of these families for all components. To each segment $s=[pq]$ we will associate two vector spaces V_{q^-} and V_{p^+} such that $V_{q^-}=V_{p^+}=\overset{\alpha_i}{V}$.

$\mathcal{S}$ being a finite set, let us choose on it a total ordering. This ordering allows us to define two vector spaces V_- and V_+ : $V_-=\bigotimes_{x\in Y\cup Z}V_{x^-}$ and $V_+=\bigotimes_{x\in Y\cup Z}V_{x^+}$ where the order in the tensor product is taken relativ to it.

Let $a,b\in Y\cup Z$ and $\xi,\eta\in\{+,-\}$, and assume that $\phi(a)=\phi(b)$, we will use as a shortcut the notation: $\epsilon(a^\xi b^\eta)=\epsilon(l(a^\xi),l(b^\eta))$.

We define the space $\Lambda_{\mathcal{S}}$ by : $\Lambda_{\mathcal{S}}=\Lambda\otimes\bigotimes_{s\in\mathcal{S}}\mathrm{End}(\mathrm{V}_{\mathrm{d(s)^-}},\mathrm{V}_{\mathrm{e(s)^+}})$.

If s is an element of $\mathcal{S}_i$ we denote by j_s the canonical injection $j_s : \Lambda \otimes \mathrm{End}(\mathrm{V}_{\mathrm{d(s)}^-}, \mathrm{V}_{\mathrm{e(s)}^+}) \hookrightarrow \Lambda_{\mathcal{S}}$.

Let us define two types of holonomy along s: $u_s \in \Lambda \otimes \mathrm{End}(\mathrm{V}_{\mathrm{d(s)}^-}, \mathrm{V}_{\mathrm{e(s)}^+})$ is defined by $u_s = u_{\varphi(s)}$, (the right handside has already been defined so that there is no risk of confusion) and $U_s \in \Lambda_{\mathcal{S}}$ is defined by $U_s = j_s(u_{\varphi(s)})$.

Definition 6 *Let a, b two points of $Y \cup Z$ such that $\phi(a) = \phi(b)$ and define the endomorphism:*

$R^{(a^\xi b^\eta)} \in \mathrm{End}(\mathrm{V}_{\mathrm{a}^\xi} \otimes \mathrm{V}_{\mathrm{b}^\eta})$ (resp $\mathrm{End}(\mathrm{V}_{\mathrm{b}^\eta} \otimes \mathrm{V}_{\mathrm{a}^\xi})$) if $s(a^\xi) \triangleleft s(b^\eta)$ (resp if $s(b^\eta) \triangleleft s(a^\xi)$) by:

$$R^{(a^\xi b^\eta)} = \begin{cases} (\alpha_1 \otimes \alpha_2)(R^{(\epsilon(a^\xi b^\eta))}) & \text{if } s(a^\xi) \triangleleft s(b^\eta) \\ P_{a^\xi b^\eta}(\alpha_1 \otimes \alpha_2)(R^{(\epsilon(a^\xi b^\eta))}) P_{a^\xi b^\eta} & \text{if } s(b^\eta) \triangleleft s(a^\xi) \end{cases}$$

Let $<$ be any fixed strict total order on Y, we define a family $\{\mathcal{R}^{(y)}\}_{y \in Y}$ of elements of $\bigotimes_{s \in \mathcal{S}} End(V_{d(s)^-}, V_{e(s)^+})$ as follows: let $\{y, y'\}$ be any pair of points of Y such that $y \sim y'$, we can always assume (otherwise we just exchange y and y') that $y < y'$,

$$\mathcal{R}^{(y)} = \begin{cases} R^{(y^- y^+)-1} \text{ if } \varphi(s(y)) \text{ is above } \varphi(s(y')) \\ R^{(y^+ y'^+)} R^{(y^- y^+)-1} R^{(y^- y'^+)-1} \text{ if } \varphi(s(y)) \text{ is under } \varphi(s(y')) \end{cases} \tag{70}$$

$$\mathcal{R}^{(y')} = \begin{cases} R^{(y'^- y'^+)-1} \text{ if } \varphi(s(y)) \text{ is above } \varphi(s(y')) \\ R^{(y'^- y'^+)-1} R^{(y^- y'^-)} R^{(y'^- y^+)}_{y'^- y^-} \text{ if } \varphi(s(y)) \text{ is under } \varphi(s(y')) \end{cases} \tag{71}$$

This definition defines completely the elements $\mathcal{R}^{(y)}$ for $y \in Y$. Similarly if z is an element of Z we will define $\mathcal{R}^{(z)} = R^{(z^- z^+)-1}$.

We have now defined the framework necessary to associate to L an element of Λ denoted W_L which generalizes the construction of Wilson loops. We denote by $<_l$ be the strict lexicographic order induced on Y by the enumeration of the connected components of L and a choice of departure point for each of these components, i.e $\overset{i}{y}_p <_l \overset{j}{y}_q$ if and only if $i < j$ or ($i = j$ and $p > q$.)

Definition 7 *Let P be a connected piece of one of the $\overset{i}{S}$. Let us choose for simplicity $P = [\overset{i}{z}_{n+1}, \overset{i}{y}_n, \overset{i}{z}_n, \cdots, \overset{i}{y}_1, \overset{i}{z}_1]$, we will denote the holonomy associated to it, by*

$$\mathcal{U}_P = \omega(P) U_{[\overset{i}{z}_{n+1} \overset{i}{y}_n]} \mathcal{R}^{(\overset{i}{y}_n)}(<_l) U_{[\overset{i}{y}_n \overset{i}{z}_n]} \mathcal{R}^{(\overset{i}{z}_n)}(<_l) \cdots \mathcal{R}^{(\overset{i}{y}_1)}(<_l) U_{[\overset{i}{y}_1 \overset{i}{z}_1]}, \tag{72}$$

where $\omega(P) = v_{\alpha_i}^{-\frac{1}{2} \sum_{x \in P \setminus \{\overset{i}{z}_{n+1}, \overset{i}{z}_1\}} \epsilon(\phi(x), \overset{i}{S})}$. We will denote by $\overset{i}{\mathcal{U}}$ the holonomy associated to the entire circle $\overset{i}{S}$. We will also define the permutation operator:

$\sigma_P = \prod_{x=\overset{i}{y}_n}^{\overset{i}{y}_1} P_{\overset{i}{z}_{n+1},x}$ *(where the order is given by the order of vertices along P)* *and* $\overset{i}{\sigma}$ *will denote* $\sigma_{\underset{S}{i}}$.

Definition 8 (Generalized Holonomies and Wilson loops) *To each link in $\Sigma \times [0,1]$ we associate an element W_L by the following procedure: let us denote by $\mathcal{W}_L$ the element*

$$\mathcal{W}_L = \mu_{\mathcal{S}} \prod_{i=1}^{p} \overset{i}{\sigma} \prod_{i=1}^{p} \overset{i}{\mathcal{U}}; \tag{73}$$

where $\mu_{\mathcal{S}} = \bigotimes_{x\in Z\setminus Z^{\partial\Sigma}} \mu_{x^+}$:

The element associated to the link L is defined by

$$W_L = tr_{\bigotimes_{x\in Z\setminus Z^{\partial\Sigma}} V_{x^+}} \mathcal{W}_L \tag{74}$$

where tr_{V_+} means the partial trace over the space V_+ after the natural identification $V_+ = V_-$.

This element satisfies important properties described by the following theorem [3]:

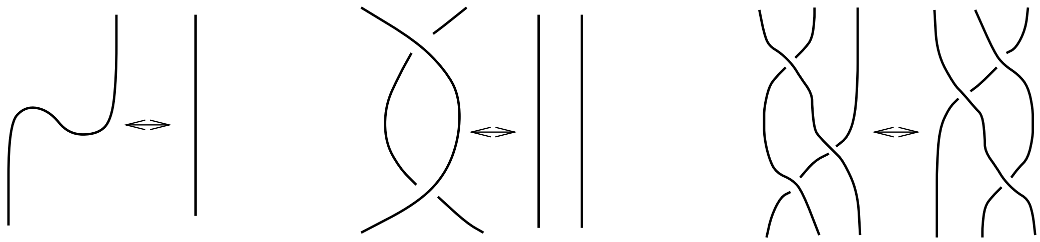
Theorem 1 *Let L be a link satisfying the set of assumptions, then W_L does not depend on the labelling of the components nor does it depend on the choice of departure points of the components. As a result W is a function on the space of links with values in $\Lambda \otimes \bigotimes_{P\in\mathcal{C}_2} End(V_{d(P)^-}, V_{e(P)^+})$. Moreover this mapping is invariant under the coaction of the gauge group at a vertex interior to the surface. If L and L' are two links , we have the morphism property $W_{L*L'} = W_L W_{L'}$.*

Our principal aim is the computation of the correlation function defined in an obvious way:

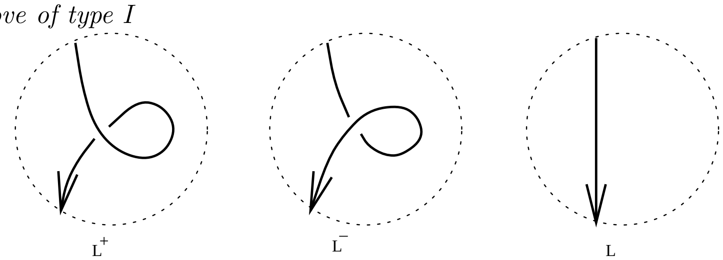
Proposition 5 (Correlation functions and Ribbons invariants) *The correlation function of the link L considered as immersed in $\Sigma \times [0,1]$ is simply defined by:*

$$< W_L >_{q-YM(\Sigma)} = \int \prod_{l\in\mathcal{L}^{int}} dh(U_l) \;\; W_L \prod_{F\in\mathcal{F}} \delta_{\partial F} \tag{75}$$

The observable associated to L will be denoted by $\widehat{W}_L = W_L \prod_{F\in\mathcal{F}} \delta_{\partial F}$. This element of Λ_{CS} depends only on the regular isotopy class of the link L, i.e it satisfies the Reidemeister moves of type 0,2,3. This fact was established in [3].



Moreover, let L be as usual a link in $\Sigma \times [0,1]$ and P the set of projected curves on Σ and let $L^{\propto\pm}$ be another link whose projection $P^{\propto\pm}$ differs from P by a move of type I



applied to a curve colored by α, we have the following relation:

$$\widehat{W}_{L^{\propto\pm}} = v_\alpha^{\pm 1}\widehat{W}_L \tag{76}$$

The expectation value of a Wilson loop on a Riemann surface can be considered as an invariant associated to a ribbon glued on the surface with the blackboard framing.

3 Computation of the correlation functions

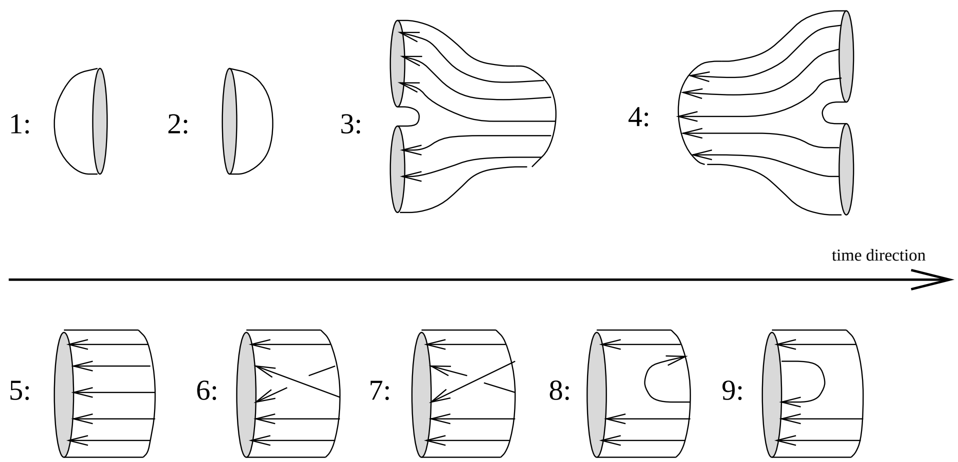
Our first aim is the computation of the invariants associated to links drawn on a closed Riemann surface embedded in S^3. To realize this program we want to decompose the computation by introducing surfaces with boundaries and links drawn on them, already called "striped surfaces", and by describing the gluing operation of the latter.

This decomposition allows us to reduce the striped surface to the gluing of the following objects, called "elementary blocks" :

1. **the cups**
2. **the caps**
3. **the (n,m)(n+m) trinions**
4. **the (n+m)(n,m) trinions**

5. **the free propagation of n strands**
6. **the propagation of n strands with one overcrossing**
7. **the propagation of n strands with one undercrossing**
8. **the (n-2)(n) creation**
9. **the (n)(n-2) annihilation**

These objects are described in the following figure



The correlation functions can be put in a more convenient form to reduce the computation to the gluing of elements associated to elementary blocks.

Lemma 1 *Let us consider an element l of $\mathcal{L}$ and an element P of $\mathcal{F}$ then*

$$P \in Past(l) \Rightarrow \delta_P \overset{\alpha}{U}(l) = \overset{\alpha}{U}(l)\delta_P.$$

Proof:

this result is a trivial consequence of the choice of ciliation and of the commutation properties developed in [2]

$\square$

From now the order induced by the orientation of the link will not be convenient anymore, prefering time ordering we will introduce the vector spaces V_{x^a} and V_{x^b} ("after" and "before") rather than V_{x^+} and V_{x^-}. Let α denote the representation associated to the circle where x is taken, then $V_{x^+} = V_{x^-} = \overset{\alpha}{V}$. If x is in $\mathcal{V}_+$, then $V_{x^a} = V_{x^b} = \overset{\alpha}{V}$ and we will introduce the canonical identification maps $id_{(V_{x^a}, V_{x^+})}$ and $id_{(V_{x^b}, V_{x^-})}$. If x is in $\mathcal{V}_-$, then $V_{x^a} = V_{x^b} = \overset{\bar{\alpha}}{V}$ and we will

introduce the canonical maps $\phi_0^{\bar{\alpha}\alpha}{}_{(V_{x^b},V_{x^+})}$ and $\psi^0_{\bar{\alpha}\alpha(V_{x^a},V_{x^-})}$. We can define a new holonomy by $U^{\#}_{[x,y]}$:

$$\begin{aligned} U^{\#}_{[x,y]} &= id_{(V_{x^a},V_{x^+})}\overset{\alpha}{U}_{[x,y]} id_{(V_{y^b},V_{y^-})}, \text{ if } x \in \mathcal{V}_+, y \in \mathcal{V}_+ \\ &= \psi^0_{\bar{\alpha}\alpha(V_{x^b},V_{x^+})}\overset{\alpha}{U}_{[x,y]}\phi_0^{\bar{\alpha}\alpha}{}_{(V_{y^a},V_{y^-})} = \overset{\bar{\alpha}}{U}_{[y,x]}, \text{ if } x \in \mathcal{V}_-, y \in \mathcal{V}_- \\ &= \psi^0_{\bar{\alpha}\alpha(V_{x^b},V_{x^+})}\overset{\alpha}{U}_{[x,y]} id_{(V_{y^b},V_{y^-})}, \text{ if } x \in \mathcal{V}_-, y \in \mathcal{V}_+ \\ &= id_{(V_{x^a},V_{x^+})}\overset{\alpha}{U}_{[x,y]}\phi_0^{\bar{\alpha}\alpha}{}_{(V_{y^a},V_{y^-})}, \text{ if } x \in \mathcal{V}_+, y \in \mathcal{V}_- \end{aligned}$$

Despite its apparent complexity, this definition has a very simple meaning. It describes the usual fact that a strand in the direction of the past coloured by a representation α can be described by a strand in the direction of the future coloured by a representation $\bar{\alpha}$.

We will denote in the following:

$$A_k = \prod_{j=1}^{Card(\mathcal{L}_{\tau_k \leq t \leq \tau_{k+1}})}((\prod_{P \in Present(l_j^k)} \delta_{\partial P})U^{\#}_{l_j^k}),$$

the elements ($(\prod_{P\in Present(l)} \delta_P U^{\#}_l)$ if l does not belong to a crossing and $(\prod_{P\in Present(l)} \delta_P U^{\#}_l U^{\#}_{l'})$ if l and l' cross themselves) will be called "square plaquettes" elements in the following.

We will also use the following permutation operator:

$$\sigma^{\tau_k} = \prod_{j=1}^{Card(\mathcal{V}_{\tau_k})} (\prod_{i=Card(\mathcal{V}_{\tau_{k+1}})}^{1} P_{(x_j^k)^a,(x_i^{k+1})^a})$$

Lemma 2 (chronologically ordered observables) *Using these definitions, the element associated to the striped surface can be put in a form which respects the ordering induced by the time order:*

$$\widehat{W}_L = v_\alpha^{\frac{1}{2}\sum_{x\in\mathcal{V}^{int}} \epsilon(x^b,x^a)} tr_{\bigotimes_{x\in\mathcal{V}^{int}} V_{x^a}}((\prod_{k=1}^{n} \sigma^{\tau_k})(\prod_{k=0}^{n} A_k)) \tag{77}$$

Proof: We begin with the ordering of the holonomies attached to the link. Using the commutation relations and the properties of the R matrix we obtain:

$$W_L = tr_{\bigotimes_{x\in\mathcal{V}^{int}} V_{x^+}}((\prod_{k=1}^{n} \prod_{j=1}^{Card(\mathcal{V}_{\tau_k})} \prod_{i=Card(\mathcal{V}_{\tau_{k+1}})}^{1} P_{(x_j^k)^+,(x_i^{k+1})^+}) \times$$

$$\times \quad (\otimes_{x\in\mathcal{V}^{int}}\mu_{x^+})(\prod_{k=0}^{n}\prod_{j=1}^{Card(\mathcal{L}_{\tau_k\leq t\leq\tau_{k+1}})}U_{l_j^k}))v_\alpha^{\frac{1}{2}(\sum_{x\in\mathcal{V}^{int}_+}\epsilon(x^-,x^+)+\sum_{x\in\mathcal{V}^{int}_-}\epsilon(x^+,x^-))}$$

now the commutation lemma gives easily:

$$(\prod_{P\in\mathcal{F}}\delta_P)(\prod_{k=0}^{n}\prod_{j=1}^{Card(\mathcal{L}_{\tau_k\leq t\leq\tau_{k+1}})}U_{l_j^k})=\prod_{k=0}^{n}\prod_{j=1}^{Card(\mathcal{L}_{\tau_k\leq t\leq\tau_{k+1}})}((\prod_{P\in Present(l_j^k)}\delta_P)U_{l_j^k}),$$

and with the definition of $U^{\#}_{l_j^k}$ we then obtain:

$$\widehat{W}_L=v_\alpha^{\frac{1}{2}\sum_{x\in\mathcal{V}^{int}}\epsilon(x^b,x^a)}tr_{\bigotimes_{x\in\mathcal{V}^{int}}V_{x^a}}((\prod_{k=1}^{n}\prod_{j=1}^{Card(\mathcal{V}_{\tau_k})}\prod_{i=Card(\mathcal{V}_{\tau_{k+1}})}^{1}P_{(x_j^k)^a,(x_i^{k+1})^a})(\prod_{k=0}^{n}A_k))$$

$\square$

This lemma leads us to a new definition of elements associated to "striped surfaces" which is based on gluing chronologically ordered elementary blocks.

Proposition 6 (Correlation functions and gluing operation) *Let us consider a "striped surface" $\Sigma+L$. Let us define an element corresponding to $\Sigma+L$ (which will be denoted by $\mathcal{A}_{\Sigma+L}$) by the following rules:*

- *if $\Sigma+L$ is an elementary block $\mathcal{B}_1$, the element of the gauge algebra associated to it is:*

$$\mathcal{A}_{\mathcal{B}_1}=\int\prod_{l\in\mathcal{L}_{]t,t'[}}dh(U_l)\prod_{j=1}^{Card(\mathcal{L}_1)}((\prod_{P\in Present(l_j^1)}\delta_P)U^{\#}_{l_j^1})v_\alpha^{\frac{1}{2}\sum_{x\in\mathcal{V}^{t'}}\epsilon(x^b,x^a)} \tag{78}$$

- *if $\Sigma+L$ is a disjoint union of N elementary blocks placed between t and t' then the element of the algebra associated to $\Sigma+L$ is obviously the product of the elements associated to each elementary block, the order between them being irrelevant because they are commuting.*

- *if there exists a time t'' between t and t' such that $\Sigma+L$ is obtained by gluing two "striped surfaces" Σ_1+L_1 and Σ_2+L_2 placed respectively between t and t'', and between t'' and t'. The element associated to $\Sigma+L$ will be defined by:*

$$\begin{aligned}\mathcal{A}_{\Sigma+L} &= \mathcal{A}_{\Sigma_1+L_1}\circ\mathcal{A}_{\Sigma_2+L_2} \\ &= \int\prod_{l\in\mathcal{L}_{t''}}dh(U_l)tr_{\bigotimes_{x\in\mathcal{V}^{t''}}V_{x^a}}(\sigma^{t''}\mathcal{A}_{\Sigma_1+L_1}\mathcal{A}_{\Sigma_2+L_2})\end{aligned} \tag{79}$$

(the canonical choice of ciliation defined for any striped surface is obviously compatible with the gluing operation)

these properties give us a new way to compute the invariants associated to links on a closed surface:

$$< W_L >_{q-YM(\Sigma)} = \mathcal{A}_{\Sigma+L}. \tag{80}$$

From now the computation of the correlation functions is reduced to the computation of elements associated to elementary blocks. After some definitions we will give the result of the explicit computation of these elements.

Definition 9 (In and Out states) *Let us consider a striped surface $\Sigma + L$. A connected component of its "In state" is a simple loop $\mathcal{C} = [x_{n+1} = x_1, x_n, \cdots, x_1]$ oriented in the inverse clockwise sense with $n+1$ strands going through it at each x_i in the direction of the past with a representation α_i.*
(a strand in the direction of the future with a representation α is reversed to the past by changing its representation in $\bar{\alpha}$). The ciliation at each x_i is chosen as in the general construction of striped surfaces. Then, choosing n other representations $(\beta_i)_{i=1,\dots,n}$ we define $\mathcal{O} \in \Lambda \otimes End(\otimes_{x\in\mathcal{C}} V_{x^b}, \mathbf{C})$ and $\mathcal{I} \in \Lambda \otimes End(\mathbf{C}, \otimes_{x\in\mathcal{C}} V_{x^a})$:
if there is at least one emerging strand through $\mathcal{C}$,

$$\begin{aligned}
\mathcal{O}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) &= v_{\beta_n}^{-1} tr_{\underset{V}{\beta_n}}(\psi^{\beta_n}_{\beta_1\alpha_1} \overset{\beta_1}{U}_{[x_1,x_2]} \psi^{\beta_1}_{\beta_2\alpha_2} \cdots \psi^{\beta_{n-1}}_{\beta_n\alpha_n} \overset{\beta_n}{U}_{[x_n,x_1]} \overset{\beta_n\alpha_1}{R^{-1}} \overset{\beta_n}{\mu}) \\
\mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) &= v_{\beta_n} tr_{\underset{V}{\beta_n}}(\overset{\beta_n}{\mu} \overset{\beta_n\alpha_1}{R} \overset{\beta_n}{U}_{[x_1,x_n]} \phi^{\beta_n\alpha_n}_{\beta_{n-1}} \cdots \phi^{\beta_2\alpha_2}_{\beta_1} \overset{\beta_1}{U}_{[x_2,x_1]} \phi^{\beta_1\alpha_1}_{\beta_n})
\end{aligned}$$

Here and in the following we will often forget the multiplicities m_i for readibility. If the connected component of this "In state" has no emerging strand we will define $\mathcal{O}$ and $\mathcal{I}$ to be:

$$\mathcal{O}(\beta_0) = W^{\beta_0}_{C^{-1}} \text{ and } \mathcal{I}(\beta_0) = W^{\beta_0}_{C} \tag{81}$$

The properties of the latter objects are described in the following lemma.

Lemma 3 *The properties of the In and Out states are generalizations of those of Wilson loops.*
Cyclicity

$$\begin{aligned}
\mathcal{O}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) &= \mathcal{O}(\beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \beta_1 \overset{\alpha_1}{x}_1 \cdots \beta_n \overset{\alpha_n}{x}_n) \\
\mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) &= \mathcal{I}(\beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \beta_1 \overset{\alpha_1}{x}_1 \cdots \beta_n \overset{\alpha_n}{x}_n)
\end{aligned} \tag{82}$$

Gauge transformation

$$\Omega(\mathcal{O}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1)) = \mathcal{O}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) \prod_{i=1}^n S(\overset{\alpha_i}{g}_{x_i})$$

$$\Omega(\mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1)) = \prod_{i=1}^n \overset{\alpha_i}{g}_{x_i} \ \mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) \tag{83}$$

Scalar product

$$\int \prod_{l \in \mathcal{C}} dh(U_l) \qquad tr_{\otimes_i V_{x_i^a}}(\sigma_{\mathcal{C}} \mathcal{O}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) \times \tag{84}$$

$$\times \mathcal{I}(\beta'_n \overset{\alpha_n}{x}_n \beta'_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta'_1 \overset{\alpha_1}{x}_1)) = \prod_{i=1}^n \delta_{\beta_i, \beta'_i}$$

Proof:

The cyclicity property is not completely obvious. We give here a detailed proof of this fact:

$$\begin{aligned}
&\mathcal{O}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) = \\
&= v_{\beta_n}^{-1} tr_{V_{\beta_n}}(\psi^{\beta_n}_{\beta_1 \alpha_1} \overset{\beta_1}{U}_{[x_1,x_2]} \psi^{\beta_1}_{\beta_2 \alpha_2} \overset{\beta_2}{U}_{[x_2,x_3]} \cdots \psi^{\beta_{n-1}}_{\beta_n \alpha_n} \overset{\beta_n}{U}_{[x_n,x_1]} \overset{\beta_n \alpha_1}{R^{-1}} \overset{\beta_n}{\mu}) \\
&= v_{\beta_n}^{-1} tr_{V_{\beta_1}}(\overset{\beta_1}{U}_{[x_1,x_2]} \psi^{\beta_1}_{\beta_2 \alpha_2} \overset{\beta_2}{U}_{[x_2,x_3]} \cdots \psi^{\beta_{n-1}}_{\beta_n \alpha_n} \overset{\beta_n}{U}_{[x_n,x_1]} \psi^{\beta_n}_{\beta_1 \alpha_1} \overset{\beta_1 \alpha_1}{R^{-1}} \overset{\beta_1}{\mu} \ v_{\alpha_2}) \\
&= v_{\beta_n}^{-1} tr_{V_{\beta_1} \otimes V'_{\beta_1}}(P_{V_{\beta_1}, V'_{\beta_1}} \overset{\beta_1}{U}_{[x_1,x_2]} \psi^{\beta'_1}_{\beta_2 \alpha_2} \overset{\beta_2}{U}_{[x_2,x_3]} \cdots \psi^{\beta_{n-1}}_{\beta_n \alpha_n} \overset{\beta_n}{U}_{[x_n,x_1]} \psi^{\beta_n}_{\beta'_1 \alpha_1} \overset{\beta'_1 \alpha_1}{R^{-1}} \overset{\beta'_1}{\mu} \ v_{\alpha_2}) \\
&= \sum_{(i),(j)} v_{\beta_n}^{-1} tr_{V_{\beta_1}}(\psi^{\beta_1}_{\beta_2 \alpha_2} b^{\beta_2}_{(i)} \overset{\beta_2}{U}_{[x_2,x_3]} \cdots \psi^{\beta_{n-1}}_{\beta_n \alpha_n} \overset{\beta_n}{U}_{[x_n,x_1]} S(a^{\beta_n}_{(j)}) \psi^{\beta_n}_{\beta_1 \alpha_1} \overset{\beta_1 \alpha_1}{R^{-1}} \overset{\beta_1}{\mu} \ v_{\alpha_2} b^{\beta_1}_{(j)} \overset{\beta_1}{U}_{[x_1,x_2]} S^2(a^{\beta_1}_{(i)})) \\
&= \sum_{(i)} v_{\beta_n}^{-1} tr_{V_{\beta_1}}(S^2(a^{\beta_1}_{(i)}) b^{\beta_1}_{(i)} \psi^{\beta_1}_{\beta_2 \alpha_2} \overset{\beta_2}{U}_{[x_2,x_3]} \cdots \psi^{\beta_{n-1}}_{\beta_n \alpha_n} \overset{\beta_n}{U}_{[x_n,x_1]} \psi^{\beta_n}_{\beta_1 \alpha_1} \overset{\beta_1}{U}_{[x_1,x_2]} \overset{\beta_1 \alpha_2}{R^{-1}} \ v_{\beta_n}) \\
&= \sum_{(i)} v_{\beta_1}^{-1} tr_{V_{\beta_1}}(\psi^{\beta_1}_{\beta_2 \alpha_2} \overset{\beta_2}{U}_{[x_2,x_3]} \cdots \psi^{\beta_{n-1}}_{\beta_n \alpha_n} \overset{\beta_n}{U}_{[x_n,x_1]} \psi^{\beta_n}_{\beta_1 \alpha_1} \overset{\beta_1}{U}_{[x_1,x_2]} \overset{\beta_1 \alpha_2}{R^{-1}} \ \mu_{\beta_1}) \\
&= \mathcal{O}(\beta_1 \overset{\alpha_1}{x}_1 \beta_n \overset{\alpha_n}{x}_n \cdots \beta_2 \overset{\alpha_2}{x}_2)
\end{aligned}$$

The gauge transformation is very simple to derive using the decomposition rules of the elements of the group and we can proove the scalar product property using simply the integration formula and the unitarity relations of Clebsch-Gordan maps.

$$\int \prod_{l \in \mathcal{C}} dh(U_l) tr_{\otimes_i V_{x_i^a}}(\sigma_{\mathcal{C}} \mathcal{O}(\beta_n m_n \overset{\alpha_n}{x}_n \beta_{n-1} m_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 m_1 \overset{\alpha_1}{x}_1) \times$$

$$\mathcal{I}(\beta'_n m'_n \overset{\alpha_n}{x}_n \beta'_{n-1} m'_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta'_1 m'_1 \overset{\alpha_1}{x}_1)) =$$

$$= \int \prod_{l\in\mathcal{C}} dh(U_l) v_{\beta_n}^{-1} v_{\beta'_n} tr_{V_{\beta_n}\otimes V_{\beta'_n}} (\psi^{\beta_n m_n}_{\beta_1\alpha_1} \overset{\beta_1}{U}_{[x_1,x_2]} \psi^{\beta_1 m_1}_{\beta_2\alpha_2} \cdots \psi^{\beta_{n-1}m_{n-1}}_{\beta_n\alpha_n} \overset{\beta'_n}{\mu} \overset{\beta'_n\alpha_1}{R} \overset{\beta_n}{U}_{[x_n,x_1]} \times$$

$$\times \overset{\beta'_n}{U}_{[x_1,x_n]} \overset{\beta_n\alpha_1}{R}^{-1} \overset{\beta_n}{\mu} \phi^{\beta'_n\alpha_n}_{\beta'_{n-1}m'_{n-1}} \cdots \phi^{\beta'_2\alpha_2}_{\beta'_1 m'_1} \overset{\beta'_1}{U}_{[x_2,x_1]} \phi^{\beta'_1\alpha_1}_{\beta'_n m'_n}) =$$

$$= \int \prod_{l\in\mathcal{C}\setminus[x_1,x_n]} dh(U_l) \frac{\delta_{\beta_n,\beta'_n}\delta_{m_n,m'_n}}{[d_{\beta_n}]} tr_{V_{\beta_n}} (\overset{\beta_n}{\mu} \psi^{\beta_n m_n}_{\beta_1\alpha_1} \overset{\beta_1}{U}_{[x_1,x_2]} \psi^{\beta_1 m_1}_{\beta_2\alpha_2} \cdots \psi^{\beta_{n-1}m_{n-1}}_{\beta_n\alpha_n} \phi^{\beta_n\alpha_n}_{\beta'_{n-1}m'_{n-1}} \cdots$$

$$\cdots \phi^{\beta'_2\alpha_2}_{\beta'_1 m'_1} \overset{\beta'_1}{U}_{[x_2,x_1]} \phi^{\beta'_1\alpha_1}_{\beta'_n m'_n}) =$$

$$= \frac{\delta_{\beta_n,\beta'_n}\delta_{m_n,m'_n}}{[d_{\beta_n}]} tr_{V_{\beta_n}} (\overset{\beta_n}{\mu}) \prod_{i=1}^{n-1} (\delta_{\beta_i,\beta'_i}\delta_{m_i,m'_i}) = \prod_{i=1}^{n} (\delta_{\beta_i,\beta'_i}\delta_{m_i,m'_i}).$$

This ends the proof of the lemma.

□

All elements associated to elementary blocks can be computed in terms of "In" and "Out" states of the latter form.

Proposition 7 *Let us give here the expression of the elements associated to the elementary blocks enumerated before:*

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{cup}} = \sum_{\beta_0} [d_{\beta_0}] \mathcal{O}(\beta_0) \tag{85}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{cap}} = \sum_{\beta_0} [d_{\beta_0}] \mathcal{I}(\beta_0) \tag{86}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{(n,m)(n+m)tri.}} = \sum_{\beta_1,\cdots,\beta_{n+m}} [d_{\beta_{n+m}}]^{-1} \mathcal{I}(\beta'_n \overset{\alpha'_n}{x'}_n \cdots \beta'_1 \overset{\alpha'_1}{x'}_1) \mathcal{I}(\beta''_n \overset{\alpha''_n}{x''}_n \cdots \beta''_1 \overset{\alpha''_1}{x''}_1) \times \tag{87}$$

$$\times \mathcal{O}(\beta_n \overset{\alpha_n}{x}_n \cdots \beta_1 \overset{\alpha_1}{x}_1) \delta_{\beta'_n,\beta''_m,\beta_{n+m},\beta_m} \prod_{k=1}^{n} \delta_{\alpha_{m+k},\alpha'_k} \prod_{k=1}^{m} \delta_{\alpha_k,\alpha''_k} \prod_{k=1}^{n-1} \delta_{\alpha_{m+k},\alpha'_k} \prod_{k=1}^{m-1} \delta_{\alpha_k,\alpha''_k}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{(n+m)(n,m)tri.}} = \sum_{\beta_1,\cdots,\beta_{n+m}} [d_{\beta_{n+m}}]^{-1} \mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \cdots \beta_1 \overset{\alpha_1}{x}_1) \mathcal{O}(\beta'_n \overset{\alpha'_n}{x'}_n \cdots \beta'_1 \overset{\alpha'_1}{x'}_1) \times \tag{88}$$

$$\times \mathcal{O}(\beta''_n \overset{\alpha''_n}{x''}_n \cdots \beta''_1 \overset{\alpha''_1}{x''}_1) \delta_{\beta'_n,\beta''_m,\beta_{n+m},\beta_m} \prod_{k=1}^{n} \delta_{\alpha_{m+k},\alpha'_k} \prod_{k=1}^{m} \delta_{\alpha_k,\alpha''_k} \prod_{k=1}^{n-1} \delta_{\alpha_{m+k},\alpha'_k} \prod_{k=1}^{m-1} \delta_{\alpha_k,\alpha''_k}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{free}} = \sum_{\beta_1,\cdots,\beta_n} \mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \cdots \beta_1 \overset{\alpha_1}{x}_1) \mathcal{O}(\beta'_n \overset{\alpha'_n}{x'}_n \cdots \beta'_1 \overset{\alpha'_1}{x'}_1) \prod_{i=1}^{n} \delta_{\beta_i,\beta'_i} \prod_{i=1}^{n} \delta_{\alpha_i,\alpha'_i} \tag{89}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{creation}} = \sum_{\beta'_1,\cdots,\beta'_n} \mathcal{I}(\beta_n \cdots \overset{\alpha_{k+2}}{x}_{k+2}\beta_k \overset{\alpha_{k-1}}{x}_{k-1}\cdots \overset{\alpha_1}{x}_1)\mathcal{O}(\beta'_n \overset{\alpha'_n}{x'}_n \cdots \beta'_1 \overset{\alpha'_1}{x'}_1) \times \tag{90}$$

$$\times \prod_{i=k+2}^{n} (\delta_{\beta_i,\beta'_i}\delta_{\alpha_i,\alpha'_i})\delta_{\beta'_{k+1},\beta_k,\beta'_{k-1}}\delta_{\alpha'_{k+1},\bar{\alpha'}_k}(v_{\beta'_k}v_{\beta_k})^{\frac{1}{2}}(\frac{[d_{\beta'_k}]}{[d_{\beta_k}]})^{\frac{1}{2}} N^{\beta'_k,m'_k}_{\beta'_{k-1}\alpha_k}\prod_{i=1}^{k-1}\delta_{\alpha_i,\alpha'_i}\prod_{i=1}^{k-2}\delta_{\beta_i,\beta'_i}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{annihil.}} = \sum_{\beta'_1,\cdots,\beta'_n} \mathcal{I}(\beta'_n \overset{\alpha'_n}{x'}_n \cdots \beta'_1 \overset{\alpha'_1}{x'}_1)\mathcal{O}(\beta_n \cdots \overset{\alpha_{k+2}}{x}_{k+2}\beta_k \overset{\alpha_{k-1}}{x}_{k-1}\cdots \overset{\alpha_1}{x}_1) \times \tag{91}$$

$$\times \prod_{i=k+2}^{n} (\delta_{\beta_i,\beta'_i}\delta_{\alpha_i,\alpha'_i})\delta_{\beta'_{k+1},\beta_k,\beta'_{k-1}}\delta_{\alpha'_{k+1},\bar{\alpha'}_k}(v_{\beta'_k}v_{\beta_k})^{-\frac{1}{2}}(\frac{[d_{\beta'_k}]}{[d_{\beta_k}]})^{\frac{1}{2}} N^{\beta'_k,m'_k}_{\beta'_{k-1}\alpha_k}\prod_{i=1}^{k-1}\delta_{\alpha_i,\alpha'_i}\prod_{i=1}^{k-2}\delta_{\beta_i,\beta'_i}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{overcross.}} = \sum_{\beta_1,\cdots,\beta_n,\beta'_k} \mathcal{I}(\beta_n \cdots \overset{\alpha_{k+1}}{x}_{k+1}\beta_k \overset{\alpha_k}{x}_k \cdots \overset{\alpha_1}{x}_1)\mathcal{O}(\beta'_n \cdots \overset{\alpha'_{k+1}}{x'}_{k+1}\beta'_k \overset{\alpha'_k}{x'}_k \cdots \overset{\alpha'_1}{x'}_1) \times$$

$$\times \prod_{i\neq k}\delta_{\beta_i,\beta'_i}\prod_{i\neq k,k+1}\delta_{\alpha_i,\alpha'_i}\frac{tr_q(\psi^{\beta_{k-2}}_{\beta_{k-1}\alpha_{k-1}}\psi^{\beta_{k-1}}_{\beta_k\alpha_k}\overset{\alpha_k\alpha_{k-1}}{\check{R}}\phi^{\beta_k\alpha_{k-1}}_{\beta'_{k-1}}\phi^{\beta'_{k-1}\alpha_k}_{\beta_{k-2}})}{[d_{\beta_{k-2}}]v^{\frac{1}{2}}_{\beta_k}v^{-\frac{1}{2}}_{\beta'_k}}\delta_{\alpha_{k+1},\alpha'_k}\delta_{\alpha_k,\alpha'_{k+1}} \tag{92}$$

$$\mathcal{A}^{\mathbf{elem}}_{\mathbf{undercross.}} = \sum_{\beta_1,\cdots,\beta_n,\beta'_k} \mathcal{I}(\beta_n \cdots \overset{\alpha_{k+1}}{x}_{k+1}\beta_k \overset{\alpha_k}{x}_k \cdots \overset{\alpha_1}{x}_1)\mathcal{O}(\beta'_n \cdots \overset{\alpha'_{k+1}}{x'}_{k+1}\beta'_k \overset{\alpha'_k}{x'}_k \cdots \overset{\alpha'_1}{x'}_1) \times$$

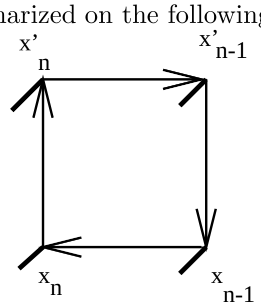
$$\times \prod_{i\neq k}\delta_{\beta_i,\beta'_i}\prod_{i\neq k,k+1}\delta_{\alpha_i,\alpha'_i}\frac{tr_q(\psi^{\beta_{k-2}}_{\beta_{k-1}\alpha_{k-1}}\psi^{\beta_{k-1}}_{\beta_k\alpha_k}\overset{\alpha_k\alpha_{k-1}}{\check{R}}{}^{-1}\phi^{\beta_k\alpha_{k-1}}_{\beta'_{k-1}}\phi^{\beta'_{k-1}\alpha_k}_{\beta_{k-2}})}{[d_{\beta_{k-2}}]v^{\frac{1}{2}}_{\beta_k}v^{-\frac{1}{2}}_{\beta'_k}}\delta_{\alpha_{k+1},\alpha'_k}\delta_{\alpha_k,\alpha'_{k+1}} \tag{93}$$

Proof:
The result for $\mathcal{A}^{elem}_{cup}$ and $\mathcal{A}^{elem}_{cap}$ is clearly given by the Boltzmann weight . The computation of the other elements need a careful description. The idea is very simple. We first absorb each link segment in the attached boltzmann weight to put each elements associated to "square plaquettes" in a same practical form where all edges of the boundary appear one and only one time and always in the same order. This form allows us to reduce the gluing of "plaquettes" to one commutation plus one integration only.
In the case of an empty square plaquette the corresponding Boltzmann weight is already in the reduced form.
For example the square plaquette element involved in the computation of a free propagation is given by:

$$\delta_{[x'_n,x_n,x_{n-1},x'_{n-1}]}\overset{\alpha_{n-1}}{U}_{[x_{n-1},x'_{n-1}]} = \sum_{\beta_{n-1}\beta'_{n-2}} [d_{\beta_{n-1}}]\lambda^{-1}_{\beta_{n-1}\alpha_{n-1}\beta_{n-2}} \times \tag{94}$$

$$\times tr_{V_{\beta_{n-1}}}(\overset{\beta_{n-1}}{\mu}\overset{\beta_{n-1}}{U}_{[x'_n,x_n]}\overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]}\phi^{\beta_{n-1}\alpha_{n-1}}_{\beta'_{n-2}}\overset{\beta'_{n-2}}{U}_{[x_{n-1},x'_{n-1}]}\psi^{\beta'_{n-2}}_{\beta_{n-1}\alpha_{n-1}}\overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]})$$

where the notations are summarized on the following figure:



In the case of a creation element for example, with a similar computation and the same notations, we have :

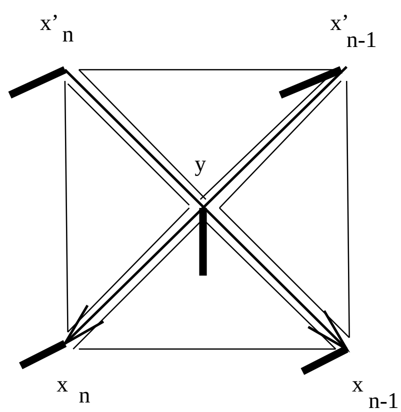
$$\delta_{[x'_n,x_n,x_{n-1},x'_{n-1}]} \overset{\alpha_{n-1}}{U}{}^{\#}_{[x'_n,x'_{n-1}]} =$$

$$= \sum_{\beta_n,\beta'_n} ([d_{\beta_n}][d_{\beta'_n}])^{\frac{1}{2}} N_{\beta_n}^{\beta'_n \alpha_n} tr_{V_{\beta_n}} (\overset{\beta_n}{\mu} \overset{\beta_n}{U}_{[x'_n,x_n]} \overset{\beta_n}{U}_{[x_n,x_{n-1}]} \overset{\beta_n}{U}_{[x_{n-1},x'_{n-1}]} \times$$

$$\times \psi^{\beta_n}_{\beta'_n \alpha_n} \overset{\beta'_n}{U}_{[x'_n,x'_{n-1}]} \psi^{\beta'_n}_{\beta_n \bar{\alpha_n}} \overset{\beta_n \bar{\alpha_n}}{R^{-1}} \lambda_{\beta_n \bar{\alpha_n} \beta'_n}) \tag{95}$$

When the square contains a crossing the reduction is less obvious and is given by the following lemma:

Lemma 4 (integration over a crossing) *With the notations of the figure:*



it was shown in our last work [3] that:

$$\int dh(U_{[x_{n-1},y]}) dh(U_{[y,x'_n]}) dh(U_{[x_n,y]}) dh(U_{[y,x'_{n-1}]}) \times$$

$$\times \delta_{[y x_n x_{n-1}]} \delta_{[y x_{n-1} x'_{n-1}]} \delta_{[y x'_{n-1} x'_n]} \delta_{[y x'_n x_n]} \overset{\alpha_n}{U}_{[x_n y x'_{n-1}]} \overset{\alpha_{n-1}}{U}_{[x_{n-1} y x'_n]} =$$

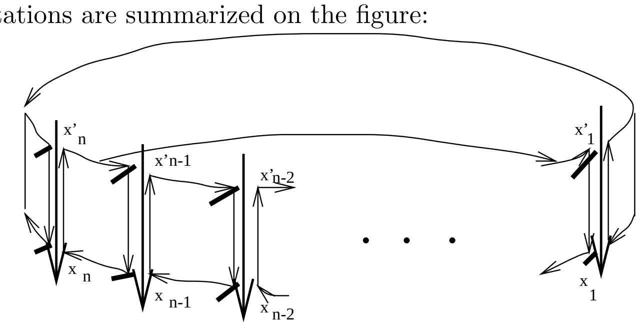
$$
= \sum_{\beta_n} [d_{\beta_n}] \left(\frac{v_{\beta'_{n-1}} v_{\beta_{n-2}}}{v_{\beta_{n-1}} v_{\beta_n}}\right)^{\frac{1}{2}} \frac{tr_q(\psi^{\beta_{k-2}}_{\beta_{k-1}\alpha_{k-1}} \psi^{\beta_{k-1}}_{\beta_k \alpha_k} \overset{\alpha_k \alpha_{k-1}}{\check{R}} \phi^{\beta_k \alpha_{k-1}}_{\beta'_{k-1}} \phi^{\beta'_{k-1}\alpha_k}_{\beta_{k-2}})}{[d_{\beta_{k-2}}]} tr_{V_{\beta_n}} (\overset{\beta_n}{\mu} \overset{\beta_n}{U}_{[x'_n x_n]} \times
$$

$$
\times \phi^{\beta_n \alpha_n}_{\beta_{n-1}} \overset{\beta_{n-1}}{U}_{[x_n x_{n-1}]} \phi^{\beta_{n-1}\alpha_{n-1}}_{\beta_{n-2}} \overset{\beta_{n-2}}{U}_{[x_{n-1} x'_{n-1}]} \psi^{\beta_{n-2}}_{\beta'_{n-1}\alpha_n} \overset{\beta'_{n-1}}{U}_{[x'_{n-1} x'_n]} \psi^{\beta'_{n-1}}_{\alpha_{n-1}\beta_n} \overset{\alpha_{n-1}\beta_n}{R'}) \qquad (96)
$$

and the analog relation for the undercrossing.

Now all square plaquettes elements are in the reduced form. Then the problem of computing the whole elementary block element is reduced to the gluing of elements associated to each of the square plaquettes given in the reduced form, i.e. a commutation + an integration. Now, let us give a careful computation in the case of $\mathcal{A}^{elem}_{free}$ and the other ones, very similar to this one, will be led to the reader.

The notations are summarized on the figure:



(**Remark:** In the following computation we forget again the multiplicities of representations in all decompositions, but we must take care of them...) Let us first describe the gluing of two square plaquettes:

$$
\int dh(U_{[x_{n-1},x'_{n-1}]}) \delta_{[x'_n,x_n,x_{n-1},x'_{n-1}]} \overset{\alpha_{n-1}}{U}_{[x_{n-1},x'_{n-1}]} \delta_{[x'_{n-1},x_{n-1},x_{n-2},x'_{n-2}]} \overset{\alpha_{n-2}}{U}_{[x_{n-2},x'_{n-2}]} =
$$

$$
= \int dh(U_{[x_{n-1},x'_{n-1}]}) \sum_{\beta_{n-1},\beta_{n-2},\beta'_{n-2},\beta'_{n-3}} [d_{\beta_{n-1}}][d_{\beta_{n-2}}] \lambda^{-1}_{\beta_{n-1}\alpha_{n-1}\beta_{n-2}} \lambda^{-1}_{\beta_{n-2}\alpha_{n-2}\beta_{n-3}} \times
$$

$$
\times tr_{V_{\beta_{n-1}}} (\overset{\beta_{n-1}}{\mu} \overset{\beta_{n-1}}{U}_{[x'_n,x_n]} \overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]} \phi^{\beta_{n-1}\alpha_{n-1}}_{\beta'_{n-2}} \overset{\beta'_{n-2}}{U}_{[x_{n-1},x'_{n-1}]} \psi^{\beta'_{n-2}}_{\beta_{n-1}\alpha_{n-1}} \overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]})
$$

$$
\times tr_{V_{\beta_{n-2}}} (\overset{\beta_{n-2}}{\mu} \overset{\beta_{n-2}}{U}_{[x'_{n-1},x_{n-1}]} \overset{\beta_{n-2}}{U}_{[x_{n-1},x_{n-2}]} \phi^{\beta_{n-2}\alpha_{n-2}}_{\beta'_{n-3}} \overset{\beta'_{n-3}}{U}_{[x_{n-2},x'_{n-2}]} \psi^{\beta'_{n-3}}_{\beta_{n-2}\alpha_{n-2}} \overset{\beta_{n-2}}{U}_{[x'_{n-2},x'_{n-1}]}) =
$$

$$
= \sum_{(i),(j)} \int dh(U_{[x_{n-1},x'_{n-1}]}) \sum_{\beta_{n-1},\beta_{n-2},\beta'_{n-2},\beta'_{n-3}} [d_{\beta_{n-1}}][d_{\beta_{n-2}}] \lambda^{-1}_{\beta_{n-1}\alpha_{n-1}\beta_{n-2}} \lambda^{-1}_{\beta_{n-2}\alpha_{n-2}\beta'_{n-3}} \times
$$

$$
\times tr_{V_{\beta_{n-1}} \otimes V_{\beta_{n-2}}} (\overset{\beta_{n-1}}{\mu} \overset{\beta_{n-1}}{U}_{[x'_n,x_n]} \overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]} \phi^{\beta_{n-1}\alpha_{n-1}}_{\beta'_{n-2}} \overset{\beta_{n-2}}{\mu} a^{\beta n-2}_{(i)} \overset{\beta_{n-2}}{\mu}^{-1} \times
$$

$$\overset{\beta'_{n-2}}{U}_{[x_{n-1},x'_{n-1}]} \overset{\beta_{n-2}}{\mu} \overset{\beta_{n-2}}{U}_{[x'_{n-1},x_{n-1}]} \overset{\beta_{n-2}}{U}_{[x_{n-1},x_{n-2}]} \phi^{\beta_{n-2}\alpha_{n-2}}_{\beta'_{n-3}} \overset{\beta'_{n-3}}{U}_{[x_{n-2},x'_{n-2}]} \psi^{\beta'_{n-3}}_{\beta_{n-2}\alpha_{n-2}} \times$$
$$\overset{\beta_{n-2}}{U}_{[x'_{n-2},x'_{n-1}]} S(a^{\beta_{n-2}}_{(j)}) \psi^{\beta'_{n-2}}_{\beta_{n-1}\alpha_{n-1}} b^{\beta_{n-1}}_{(i)} b^{\beta_{n-1}}_{(j)} \overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]}) =$$
$$= \sum_{(i),(j)} \sum_{\beta_{n-1},\beta_{n-2},\beta'_{n-2},\beta'_{n-3}} [d_{\beta_{n-1}}][d_{\beta_{n-2}}] \lambda^{-1}_{\beta_{n-1}\alpha_{n-1}\beta_{n-2}} \lambda^{-1}_{\beta_{n-2}\alpha_{n-2}\beta'_{n-3}} \frac{\delta_{\beta_{n-2},\beta'_{n-2}}}{[d_{\beta_{n-2}}]} \times$$
$$\times tr_{V_{\beta_{n-1}}}(\overset{\beta_{n-1}}{\mu} \overset{\beta_{n-1}}{U}_{[x'_n,x_n]} \overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]} \phi^{\beta_{n-1}\alpha_{n-1}}_{\beta_{n-2}} \overset{\beta_{n-2}}{U}_{[x_{n-1},x_{n-2}]} \phi^{\beta_{n-2}\alpha_{n-2}}_{\beta'_{n-3}} \overset{\beta'_{n-3}}{U}_{[x_{n-2},x'_{n-2}]} \times$$
$$\times \psi^{\beta'_{n-3}}_{\beta_{n-2}\alpha_{n-2}} \overset{\beta_{n-2}}{U}_{[x'_{n-2},x'_{n-1}]} S(a^{\beta_{n-2}}_{(j)}) S^2(a^{\beta n-2}_{(i)}) \psi^{\beta_{n-2}}_{\beta_{n-1}\alpha_{n-1}} b^{\beta_{n-1}}_{(i)} b^{\beta_{n-1}}_{(j)} \overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]}) =$$
$$= \sum_{\beta_{n-1},\beta_{n-2},\beta'_{n-3}} [d_{\beta_{n-1}}] \lambda^{-1}_{\beta_{n-1}\alpha_{n-1}\beta_{n-2}} \lambda^{-1}_{\beta_{n-2}\alpha_{n-2}\beta'_{n-3}} tr_{V_{\beta_{n-1}}}(\overset{\beta_{n-1}}{\mu} \overset{\beta_{n-1}}{U}_{[x'_n,x_n]} \overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]} \times$$
$$\phi^{\beta_{n-1}\alpha_{n-1}}_{\beta_{n-2}} \overset{\beta_{n-2}}{U}_{[x_{n-1},x_{n-2}]} \phi^{\beta_{n-2}\alpha_{n-2}}_{\beta'_{n-3}} \overset{\beta'_{n-3}}{U}_{[x_{n-2},x'_{n-2}]} \psi^{\beta'_{n-3}}_{\beta_{n-2}\alpha_{n-2}} \overset{\beta_{n-2}}{U}_{[x'_{n-2},x'_{n-1}]} \times$$
$$\times \psi^{\beta_{n-2}}_{\beta_{n-1}\alpha_{n-1}} \overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]})$$

In the same way we can glue $n-1$ Boltzmann weights and links. As a result we have obviously:

$$\int \prod_{i=1}^{n-1} dh(U_{[x_i,x'_i]}) \prod_{i=1}^{n-1} (\delta_{[x'_{i+1},x_{i+1},x_i,x'_i]} \overset{\alpha_i}{U}_{[x_i,x'_i]}) =$$
$$= \sum_{\beta_{n-1},\cdots,\beta_2,\beta_1} [d_{\beta_{n-1}}] \prod_{i=2}^{n-1} \lambda^{-1}_{\beta_i\alpha_i\beta_{i-1}} \lambda^{-1}_{\beta_1\alpha_1\beta'_n} tr_{V_{\beta_{n-1}}}(\overset{\beta_{n-1}}{\mu} \overset{\beta_{n-1}}{U}_{[x'_n,x_n]} \overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]} \phi^{\beta_{n-1}\alpha_{n-1}}_{\beta_{n-2}} \cdots$$
$$\cdots \phi^{\beta_1\alpha_1}_{\beta'_n} \overset{\beta'_n}{U}_{[x_1,x'_1]} \psi^{\beta'_n}_{\beta_1\alpha_1} \cdots \psi^{\beta_{n-2}}_{\beta_{n-1}\alpha_{n-1}} \overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]})$$

The computation of $\mathcal{A}^{elem}_{free}$ can be achieved by the gluing of the last Boltzmann weight to obtain the cylinder with n strands.This Boltzmann weight must be changed to the square plaquette element corresponding to crossings,... in the computations of the other blocks elements.

$$\int \prod_{i=1}^{n} dh(U_{[x_i,x'_i]}) \prod_{i=1}^{n} (\delta_{[x'_{i+1},x_{i+1},x_i,x'_i]} \overset{\alpha_i}{U}_{[x_i,x'_i]}) =$$
$$= \int dh(U_{[x_n,x'_n]}) dh(U_{[x_1,x'_1]}) \times$$
$$(\sum_{\beta_n,\beta'_{n-1}} [d_{\beta_n}] \lambda^{-1}_{\beta_n\alpha_n\beta'_{n-1}} tr_{V_{\beta_n}}(\overset{\beta_n}{\mu} \overset{\beta_n}{U}_{[x_1,x_n]} \phi^{\beta_n\alpha_n}_{\beta'_{n-1}} \overset{\beta'_{n-1}}{U}_{[x_n,x'_n]} \psi^{\beta'_{n-1}}_{\beta_n\alpha_n} \overset{\beta_n}{U}_{[x'_n,x'_1]} \overset{\beta_n}{U}_{[x'_1,x_1]})) \times$$

$$
\times(\sum_{\beta_{n-1},\cdots,\beta_2,\beta_1}[d_{\beta_{n-1}}]\prod_{i=2}^{n-1}\lambda^{-1}_{\beta_i\alpha_i\beta_{i-1}}\lambda^{-1}_{\beta_1\alpha_1\beta'_n}tr_{V_{\beta_{n-1}}}(\overset{\beta_{n-1}}{\mu}\overset{\beta_{n-1}}{U}_{[x'_n,x_n]}\overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]}\phi^{\beta_{n-1}\alpha_{n-1}}_{\beta_{n-2}}\cdots
$$

$$
\cdots\phi^{\beta_1\alpha_1}_{\beta'_n}\overset{\beta'_n}{U}_{[x_1,x'_1]}\psi^{\beta'_n}_{\beta_1\alpha_1}\cdots\psi^{\beta_{n-2}}_{\beta_{n-1}\alpha_{n-1}}\overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]})) =
$$

$$
= \int dh(U_{[x_n,x'_n]})dh(U_{[x_1,x'_1]})\sum_{\beta'_{n-1},\beta_n,\cdots,\beta_2,\beta'_1}[d_{\beta_{n-1}}][d_{\beta_n}]\prod_{i=2}^{n-1}\lambda^{-1}_{\beta_i\alpha_i\beta_{i-1}}\lambda^{-1}_{\beta_1\alpha_1\beta'_n}\lambda^{-1}_{\beta_n\alpha_n\beta'_{n-1}}\times
$$

$$
tr_{V_{\beta_n}\otimes V_{\beta_{n-1}}}(\overset{\beta_n}{\mu}\overset{\beta_n}{U}_{[x_1,x_n]}\phi^{\beta_n\alpha_n}_{\beta'_{n-1}}\overset{\beta_{n-1}}{\mu}a^{\beta_{n-1}}_{(j)}\overset{\beta_{n-1}}{\mu}^{-1}\overset{\beta'_{n-1}}{U}_{[x_n,x'_n]}\overset{\beta_{n-1}}{\mu}\overset{\beta_{n-1}}{U}_{[x'_n,x_n]}\overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]}\times
$$

$$
\times\phi^{\beta_{n-1}\alpha_{n-1}}_{\beta_{n-2}}\cdots\overset{\beta_1}{U}_{[x_1,x_n]}b^{\beta_1}_{(i)}\phi^{\beta_1\alpha_1}_{\beta'_n}\psi^{\beta'_{n-1}}_{\beta_n\alpha_n}b^{\beta_n}_{(j)}\overset{\beta_n}{U}_{[x'_n,x'_1]}\overset{\beta_n}{U}_{[x'_1,x_1]}\overset{\beta'_n}{U}_{[x_1,x'_1]}S(a^{\beta'_n}_{(i)})\psi^{\beta'_n}_{\beta_1\alpha_1}\cdots
$$

$$
\cdots\psi^{\beta_{n-2}}_{\beta_{n-1}\alpha_{n-1}}\overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]})) =
$$

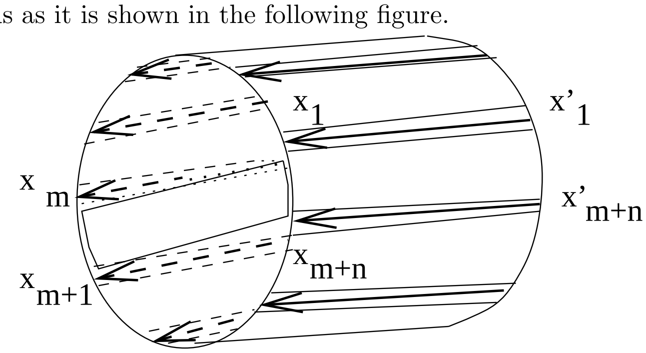
$$
= \sum_{\beta_n,\cdots,\beta_1}[d_{\beta_n}]\prod_{i=1}^{n}v^{-\frac{1}{2}}_{\alpha_i}tr_{V_{\beta_n}}(\overset{\beta_n}{\mu}\overset{\beta_n}{U}_{[x_1,x_n]}\phi^{\beta_n\alpha_n}_{\beta_{n-1}}\overset{\beta_{n-1}}{U}_{[x_n,x_{n-1}]}\phi^{\beta_{n-1}\alpha_{n-1}}_{\beta_{n-2}}\cdots\overset{\beta_1}{U}_{[x_1,x_n]}b^{\beta_1}_{(i)}\phi^{\beta_1\alpha_1}_{\beta_n}\times
$$

$$
\times\overset{\beta_n}{\mu}^{-1}S(a^{\beta_n}_{(i)}))tr_{V_{\beta_{n-1}}}(S^2(a^{\beta_{n-1}}_{(j)})\psi^{\beta_{n-1}}_{\beta_n\alpha_n}b^{\beta_n}_{(j)}\overset{\beta_n}{U}_{[x'_n,x'_1]}\psi^{\beta'_n}_{\beta_1\alpha_1}\cdots\psi^{\beta_{n-2}}_{\beta_{n-1}\alpha_{n-1}}\overset{\beta_{n-1}}{U}_{[x'_{n-1},x'_n]}) =
$$

$$
= \sum_{\beta_1,m_1,\cdots,\beta_n,m_n}\mathcal{I}(\beta_n m_n \overset{\alpha_n}{x}_n\cdots\beta_1 m_1\overset{\alpha_1}{x}_1)\mathcal{O}(\beta_n m_n\overset{\alpha_n}{x}_n\cdots\beta_1 m_1\overset{\alpha_1}{x'}_1)
$$

This concludes the computation of $\mathcal{A}^{elem}_{overcross}$, $\mathcal{A}^{elem}_{undercross}$, $\mathcal{A}^{elem}_{creation}$, $\mathcal{A}^{elem}_{annihil}$, and $\mathcal{A}^{elem}_{free}$.

The computations of $\mathcal{A}^{elem}_{(n,m)(n+m)tri.}$ and $\mathcal{A}^{elem}_{(n,m)(n+m)tri.}$ need one more step. It uses naturally the expression of $\mathcal{A}^{elem}_{free}$ as a basic object. Indeed we compute the element associated to the trinion by gluing one more plaquette to the cylinder with n strands as it is shown in the following figure.



the computation is realized by the usual techniques:

$$\mathcal{A}_{(n,m)(n+m)tri.}=\int dh(U_{[x_1,x_{m+n}]})dh(U_{[x_{m+1},x_m]})\delta_{[x_1,x_m,x_{m+1},x_{m+n}]}\times$$

$$\times\sum_{\beta_1,\cdots,\beta_{m+n}}\mathcal{I}(\beta_{m+n}\overset{\alpha_{n+m}}{x}_{n+m}\cdots\beta_1\overset{\alpha_1}{x}_1)\mathcal{O}(\beta_{m+n}\overset{\alpha_{n+m}}{x'}_{n+m}\cdots\beta_1\overset{\alpha_1}{x'}_1)=$$

$$\int dh(U_{[x_1,x_{m+n}]})dh(U_{[x_{m+1},x_m]})\sum_{\beta'_{m+n}}[d_{\beta'_{m+n}}]v^{-1}_{\beta'_{m+n}}tr_{V_{\beta'_{m+n}}}(\overset{\beta'_{m+n}}{\mu}\overset{\beta'_{m+n}}{U}_{[x_{m+n},x_1]}\overset{\beta'_{m+n}}{U}_{[x_1,x_m]}\times$$

$$\times\overset{\beta'_{m+n}}{U}_{[x_m,x_{m+1}]}\overset{\beta'_{m+n}}{U}_{[x_{m+1},x_{m+n}]})\sum_{\beta_{m+n},\cdots,\beta_1}tr_{V_{\beta_{n+m}}}(\overset{\beta_{n+m}}{\mu}\overset{\beta_{m+n}\alpha_1}{R}\overset{\beta_{n+m}}{U}_{[x_1,x_{n+m}]}\phi^{\beta_{m+n}\alpha_{m+n}}_{\beta_{m+n-1}}\cdots$$

$$\cdots\overset{\beta_1}{U}_{[x_2,x_1]}\phi^{\beta_1\alpha_1}_{\beta_{m+n}})\ \mathcal{O}(\beta_{m+n}\overset{\alpha_{n+m}}{x'}_{n+m}\cdots\beta_1\overset{\alpha_1}{x'}_1)=$$

$$=\int dh(U_{[x_1,x_{m+n}]})dh(U_{[x_{m+1},x_m]})\sum_{\beta'_{m+n},\beta_{m+n},\cdots,\beta_1}[d_{\beta'_{m+n}}]v^{-1}_{\beta'_{m+n}}\times$$

$$\times tr_{V_{\beta'_{m+n}}\otimes V_{\beta_{n+m}}}(\overset{\beta'_{m+n}\beta_{n+m}}{\mu}\overset{\beta_{n+m}}{\mu}\overset{\beta_{n+m}\alpha_1}{R}S^{-1}(b^{\beta_{m+n}}_{(j)})\overset{\beta_{n+m}}{\mu}^{-1}(\overset{\beta'_{m+n}}{U}_{[x_{m+n},x_1]})\overset{\beta_{n+m}}{\mu}\overset{\beta_{n+m}}{U}_{[x_1,x_{n+m}]})\times$$

$$\times a^{\beta'_{n+m}}_{(j)}a^{\beta_{n+m}}_{(i)}\overset{\beta'_{m+n}}{U}_{[x_1,x_m]}\phi^{\beta_{m+n}\alpha_{m+1}}_{\beta_{m+n-1}}a^{\beta_{n+m-1}}_{(l)}\cdots\phi^{\beta_{m+1}\alpha_{m+1}}_{\beta_m}S^{-1}(b^{\beta_m}_{(k)})\overset{\beta_{m+1}}{\mu}^{-1}\times$$

$$\times(\overset{\beta'_{m+n}}{U}_{[x_m,x_{m+1}]}\overset{\beta_{m+1}}{\mu}\overset{\beta_{m+1}}{U}_{[x_{m+1},x_m]})a^{\beta'_{n+m}}_{(k)}\overset{\beta'_{m+n}}{U}_{[x_{m+1},x_{m+n}]}b^{\beta'_{m+n}}_{(i)}S(b^{\beta'_{m+n}}_{(l)})\phi^{\beta_m\alpha_m}_{\beta_{m-1}}\cdots$$

$$\cdots\overset{\beta_1}{U}_{[x_2,x_1]}\phi^{\beta_1\alpha_1}_{\beta_{m+n}})\mathcal{O}(\beta_{m+n}\overset{\alpha_{n+m}}{x'}_{n+m}\cdots\beta_1\overset{\alpha_1}{x'}_1)=$$

$$=\sum_{\beta'_{m+n},\beta_{m+n},\cdots,\beta_1}v_{\beta_m}[d_{\beta_{m+n}}]^{-1}\delta_{\beta_{m+n},\beta_m,\beta'_{m+n}}\times$$

$$\times tr_{V_{\beta_m}}(\overset{\beta_m\beta_{n+m}}{\mu}\overset{\beta_m\alpha_1}{\mu}\overset{}{R}\overset{\beta_m}{U}_{[x_1,x_m]}\phi^{\beta_m\alpha_m}_{\beta_{m-1}}\cdots\overset{\beta_1}{U}_{[x_2,x_1]}\phi^{\beta_1\alpha_1}_{\beta_m})\times$$

$$\times tr_{V_{\beta_{m+n}}}(\overset{\beta_{m+n}}{\mu}a^{\beta_{n+m}}_{(i)}\phi^{\beta_{m+n}\alpha_{m+n}}_{\beta_{m+n-1}}a^{\beta_{n+m-1}}_{(l)}\cdots\phi^{\beta_{m+1}\alpha_{m+1}}_{\beta_{m+n}}\overset{\beta_{m+n}}{U}_{[x_{m+1},x_{m+n}]}S(b^{\beta_{m+n}}_{(l)})b^{\beta_{m+n}}_{(i)})\times$$

$$\times\mathcal{O}(\beta_{m+n}\overset{\alpha_{n+m}}{x'}_{n+m}\cdots\beta_1\overset{\alpha_1}{x'}_1)=$$

$$=\sum_{\beta_1,\cdots,\beta_{n+m}}[d_{\beta_{n+m}}]^{-1}\mathcal{I}(\beta'_n\overset{\alpha'_n}{x'}_n\cdots\beta'_1\overset{\alpha'_1}{x'}_1)\mathcal{I}(\beta''_n\overset{\alpha''_n}{x''}_n\cdots\beta''_1\overset{\alpha''_1}{x''}_1)\times$$

$$\times\mathcal{O}(\beta_n\overset{\alpha_n}{x}_n\cdots\beta_1\overset{\alpha_1}{x}_1)\delta_{\beta'_n,\beta''_m,\beta_{n+m},\beta_m}\prod_{k=1}^{n}\delta_{\alpha_{m+k},\alpha'_k}\prod_{k=1}^{m}\delta_{\alpha_k,\alpha''_k}\prod_{k=1}^{n-1}\delta_{\alpha_{m+k},\alpha'_k}\prod_{k=1}^{m-1}\delta_{\alpha_k,\alpha''_k}$$

$$=\sum_{\beta_1,\cdots,\beta_{n+m}}[d_{\beta_{n+m}}]^{-1}\mathcal{I}(\beta_n\overset{\alpha_n}{x}_n\cdots\beta_1\overset{\alpha_1}{x}_1)\mathcal{O}(\beta'_n\overset{\alpha'_n}{x'}_n\cdots\beta'_1\overset{\alpha'_1}{x'}_1)\times$$

$$
\times\mathcal{O}(\beta''_n \overset{\alpha''_n}{x''}_n \cdots \beta''_1 \overset{\alpha''_1}{x''}_1)\delta_{\beta'_n,\beta''_m,\beta_{n+m},\beta_m} \prod_{k=1}^{n} \delta_{\alpha_{m+k},\alpha'_k} \prod_{k=1}^{m} \delta_{\alpha_k,\alpha''_k} \prod_{k=1}^{n-1} \delta_{\alpha_{m+k},\alpha'_k} \prod_{k=1}^{m-1} \delta_{\alpha_k,\alpha''_k}
$$

This ends the proof of the Theorem.

□

These results can be easily generalized to the case where q is a root of unity. It suffices to realize the following replacements:
The expression for the In state becomes

$$
\begin{aligned}
&\mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \beta_{n-1} \overset{\alpha_{n-1}}{x}_{n-1} \cdots \beta_1 \overset{\alpha_1}{x}_1) = \\
&= v_{\beta_n} tr_{V_{\beta_n}} (S(A) \overset{\beta_n}{\mu}\ \overset{\alpha_1}{\theta_l^{(1)}} \overset{\beta_n}{\theta_l^{(2)}}\ \overset{\beta_n \alpha_1}{R}\ \overset{\beta_n}{\theta_k^{-1(1)}} \overset{\alpha_1}{\theta_k^{-1(2)}} \overset{\beta_n}{U}_{[x_1,x_n]} S(\overset{\beta_n}{\theta_m^{(1)}}) \overset{\beta_n}{A} \overset{\beta_n}{\theta_m^{(2)}} \otimes \overset{\alpha_n}{\theta_m^{(3)}} \phi^{\beta_n \alpha_n}_{\beta_{n-1}} \cdots \\
&\cdots \phi^{\beta_2 \alpha_2}_{\beta_1} \overset{\beta_1}{U}_{[x_2,x_1]} S(\overset{\beta_1}{\theta_i^{(1)}}) S(\overset{\beta_1}{\theta_j^{(1)}}) \overset{\beta_1}{A} \overset{\beta_1}{\theta_j^{(2)}} \otimes \overset{\alpha_1}{\theta_j^{(3)}} \phi^{\beta_1 \alpha_1}_{\beta_n} \overset{\beta_n}{\theta_i^{(2)}} \overset{\beta_n}{B} S(\overset{\beta_n}{\theta_i^{(3)}}) S(\overset{\beta_n}{\theta_k^{-1(3)}}) S(\overset{\beta_n}{\theta_l^{(3)}}))
\end{aligned}
$$

and the analog formula for the Out state. The $6-j$ associated to the crossing is changed to obtain an intertwiner.
The other objects remain unchanged and the summations are restricted to physical representations only.

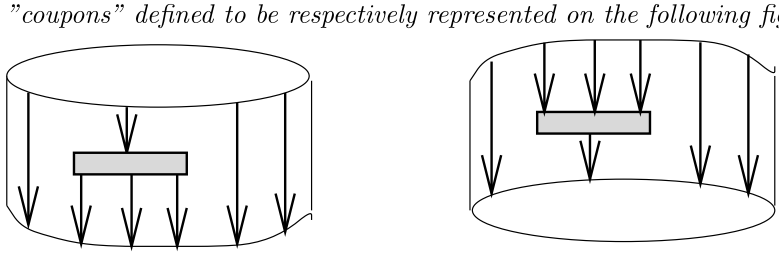
We have the following corollary:

Proposition 8 *If L is a link without boundaries in $D \times [0,1]$ we have for any value of q:*

$$
\frac{< W_L >_{q-YM(S^2)}}{< 1 >_{q-YM(S^2)}} = RT_{\mathcal{U}_q(\mathcal{G})}(L) \tag{97}
$$

where RT is the Reshetikhin-Turaev's quantum invariant of coloured links. This result has already been shown in [3]. and generally if Σ is a closed surface, the invariant associated to $\Sigma + L$ is simply a generalization to the case of a surface of the Reshetikhin-Kirillov invariant in the shadow world [11]. This theorem can be considered as a proof of the equivalence of Invariants arising from Chern-Simons theory and Reshetikhin-Turaev quantum invariants.

Definition 10 *We can also generalize our invariant to admit other objects called "coupons" defined to be respectively represented on the following figure:*

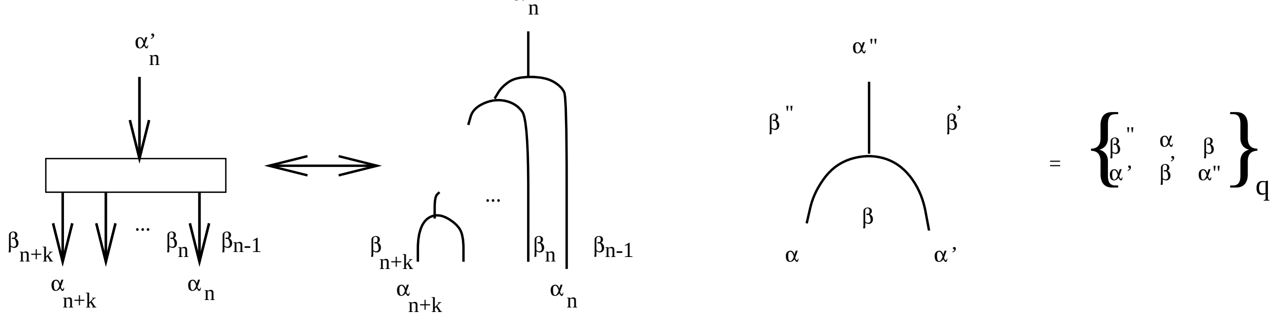


and which expressions are given by:

$$\mathcal{A}_{(n+k)(n)coupon} = \sum \mathcal{I}(\beta_{n+k} \overset{\alpha_{n+k}}{x}_{n+k} \cdots \beta_1 \overset{\alpha_1}{x}_1) \times Shadow(coupon) \times \mathcal{O}(\beta'_n \overset{\alpha'_n}{x'}_n \cdots \beta_1 \overset{\alpha_1}{x}_1)$$

$$\mathcal{A}_{(n)(n+k)coupon} = \sum \mathcal{I}(\beta_n \overset{\alpha_n}{x}_n \cdots \beta_1 \overset{\alpha_1}{x}_1) \times Shadow(coupon) \times \mathcal{O}(\beta'_{n+k} \overset{\alpha'_{n+k}}{x'}_{n+k} \cdots \beta_1 \overset{\alpha_1}{x}_1)$$

with $Shadow(coupon)$ *being defined as usual by the following rule:*



4 A new description of invariants of three manifolds

In this chapter q will be a root of unity.

4.1 Heegaard splitting and surgery of 3-manifolds

In the following $\mathcal{M}$ is a compact orientable 3-manifold given by a simplicial complex K.Let us recall standard definitions that can be found in [13].

Definition 11 *A* **canonical region** $\mathcal{R}$ *of* $\mathcal{M}$ *is a region within which there are p non intersecting 2-cells* $(E_i)_{i=1\cdots p}$*(the* **canonical cells***) with boundaries* e_i *(the* **canonical curves***) on the boundary* $\mathcal{L}$ *of* $\mathcal{R}$ *such that we obtain a 3-cell by cutting* $\mathcal{R}$ *at each* (E_i)*. A surface* $\mathcal{L}$ *is said to be a* **canonical surface** *of a 3-manifold* $\mathcal{M}$ *if it satisfies these conditions:*

- $\mathcal{L}$ *is a subcomplex of* $\mathcal{M}$ *and is a compact, connected 2-dimensional manifold*
- $\mathcal{M} = \mathcal{R}_1 + \mathcal{L} + \mathcal{R}_2$ *with* $\mathcal{R}_1, \mathcal{R}_2$ *canonical regions and* $\mathcal{L} = \partial\mathcal{R}_1 = \partial\mathcal{R}_2$

such a decomposition is called **canonical decomposition***. It is important to recall that, if g is the genus of $\mathcal{L}$, in this case, $\mathcal{R}_1$ and $\mathcal{R}_2$ are homeomorphic to a genus g handlebody.*

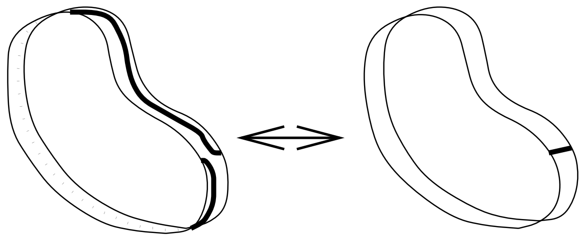
Remark: It is easy to give, for each $\mathcal{M}$, at least one canonical decomposition. To this aim, let us consider $\{A_0^i\}$, $\{A_1^j\}$, $\{A_2^k\}$, $\{A_3^l\}$ the sets of $0-, 1-, 2-, 3-$cells of K. Let $\{B_1^j\}$, $\{B_2^k\}$, $\{B_3^l\}$ be respectively the middle of $\{A_1^j\}$, $\{A_2^k\}$, $\{A_3^l\}$. The complex K' obtained by adding the B_1s, the B_2s and the B_3s to the vertices of K is called the **first derived complex of K**, its 3-simplexes are of the form $(A_0B_1B_2B_3)$. The **second derived complex of K**, denoted K" is the complex complex generated from K' by adding the vertices C_1s, C_2s, C_3s middle of the $1-, 2-, 3-$simplexes of K'. Let us denote by $\mathcal{R}_1$ the set of all $3-$simplexes of K'' of the type $(A_0C_1C_2C_3)$ or $(B_1C_1C_2C_3)$, by $\mathcal{R}_2$ the set of all $3-$simplexes of K" of the type $(B_2C_1C_2C_3)$ or $(B_3C_1C_2C_3)$ and by $\mathcal{L}$ the common frontier of $\mathcal{R}_1$ and $\mathcal{R}_2$. If we call G (resp.$G^\star$) the linear graph generated by the 1-simplexes of K (resp. its dual) we can see that $\mathcal{R}_1$ and $\mathcal{R}_2$ are respectively $K''-$neighbourhood of G and $G^\star$. Then we have that $\mathcal{M} = \mathcal{R}_1 + \mathcal{L} + \mathcal{R}_2$ is a canonical decomposition, it is called the **canonical decomposition derived from the triangulation**. It is easy to check that for each canonical decomposition, there exits a triangulation of $\mathcal{M}$ such that the decomposition is in fact the canonical decomposition derived from the triangulation.

Definition 12 Heegaard Splitting
A Heegaard splitting of a 3-manifold $\mathcal{M}$ is a set (g, f) where g is a non negative integer and f is a diffeomorphism of a genus g surface $\mathcal{L}_g$ such that $\mathcal{M}$ is the manifold obtained by gluing two copies of the handlebody $\mathcal{T}_g$ (the interior of $\mathcal{L}_g$) along their boundaries after having acted on one of them by f:

$$\mathcal{M} = \mathcal{T}_g \#_f \mathcal{T}_g$$

A Heegaard diagram is a set $(\mathcal{L}, (e_i)_{i=1\cdots g}, (f_j)_{j=1\cdots g})$ where $\mathcal{L}$ is a compact connected 2-dimensional manifold of genus g and $(e_i)_{i=1\cdots g}$ (resp.$(f_j)_{j=1\cdots g})$) are canonical curves of $\mathcal{R}_1$, the region interior to $\mathcal{L}$ (resp. canonical curves of $\mathcal{R}_2$ the exterior of $\mathcal{L}$).This data is sufficient to reconstruct an element f of $Diff(\mathcal{L})$ such that $f(e_j) = f_j$. Two Heegaard diagrams are said to be equivalent if they describe homeomorphic 3-manifolds. Let $({f'}_i)_{i=1...g}$ be g other canonical curves in $\mathcal{L}$ such that $(\mathcal{L}, (e_i)_{i=1\cdots g}, ({f'}_j)_{j=1\cdots g})$ is a Heegaard diagram of the sphere S^3 then $(\mathcal{L}, (e_i)_{i=1\cdots g}, (f_j)_{j=1\cdots g}, ({f'}_j)_{j=1\cdots g})$ is said to be an augmented Heegaard diagram.

We must recall that any element of the moduli space of a surface can be written as the composition of Dehn twists. A Dehn twist can be described by the following replacement of a regular neighbourhood of the corresponding curve:



There is an important theorem due to Singer [13] describing the relation between equivalent Heegaard diagrams.

Definition 13 Singer's elementary moves *Let us describe a set of elementary moves on the Heegaard diagrams:*

type 0: trivial moves

- *replace a curve by another curve isotopic to it, or to its inverse, or reembedded the canonical surface $\mathcal{L}$ in a different way in S^3.*

type 1: solid handlebodies diffeomorphisms

- *replace one canonical curve of the set $(e_i)_{i=1\cdots g}$ (resp. $(f_j)_{j=1\cdots g}$) by the composition of this curve with another one in this set.*
- *making a Dehn twist along one of the e_is.*

type 2: $g \rightarrow g+1$ moves

- *add a handle to $\mathcal{L}$ and define e_{p+1} (resp. f_{p+1}) to be the $a-$cycle (resp. the $b-$cycle) of this handle, or erase a handle with cycles for which e_{p+1} is the $a-$cycle (resp. f_{p+1} is the $b-$cycle).*

Then we have the following classification theorem [13]:

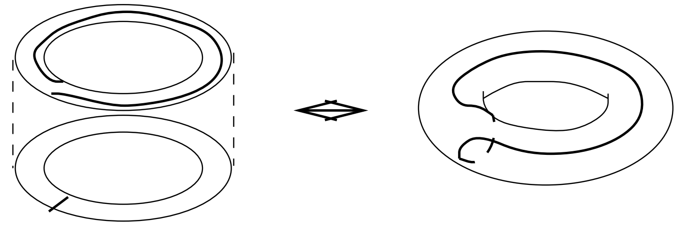
Proposition 9 (Singer's Theorem) *If the diagrams D and D', related by a finite number of Singer's moves, give rise to the manifolds $\mathcal{M}$ and $\mathcal{M}'$ then $\mathcal{M}$ and $\mathcal{M}'$ are homeomorphic. Conversely, if D and D' are any two Heegaard diagrams whatsoever arising from a manifold $\mathcal{M}$ then D and D' are related by a finite number of Singer's moves.*

A more generally used description of three manifolds is "the surgery presentation". Let us recall some facts about this description [14].

Definition 14 (Surgery presentation of 3-manifolds) *Let $(R, r) = \cup_{i=1}^{n}(R_i, r_i)$ be a framed link in the oriented sphere S^3. We can define a manifold $\mathcal{M}$ by "surgery" from (R, r) using the following procedure:*
remove from S^3 pairwise disjoint tubular neighbourhoods V_i of the curves R_i

and resew them identifying a meridian z_i in ∂V_i with a curve $y_i \in \partial(S^3 \setminus V_i^{int})$ which links R_i exactly r_i times.
Moreover, every 3-manifold $\mathcal{M}$ can be obtained from a certain framed link by this procedure [14].

It is relatively easy to relate the Heegaard and Surgery points of view [15]. Let us consider a Heegaard diagram based on a gluing diffeomorphism f described in terms of Dehn twists of the surface. We first remark that splitting S^3 along $\mathcal{L}$ then doing a Dehn twist along a certain curve and resewing the handlebody is equivalent to do a surgery along the ribbon glued on the surface along this curve as it can be seen on the figure:



Let $(R_i)_{i=1..n}$ be a set of ribbons trivially embedded on the surface $\mathcal{L}$, f_i the corresponding Dehn twists. We want also define the framed link L defined to be the set of ribbons $(R_i \times \epsilon_i)_{i=1\cdots n}$ for $0 \leq \epsilon_1 \leq \cdots \leq \epsilon_n \leq 1$. We consider a partition of S^3 in three pieces: $\mathcal{L} \times [0,1]$, the handlebody $\mathcal{H}_g$ interior to $\mathcal{L} \times \{0\}$ and the handlebody $\mathcal{H}_g^{'}$ exterior to $\mathcal{L} \times \{1\}$. Let us consider the manifold $\mathcal{M}(f_1, f_2, \cdots, f_n)$ obtained by gluing the manifolds $\mathcal{H}_g$, $\mathcal{H}_g^{'}$, $(\mathcal{L} \times [\epsilon_{i-1}, \epsilon_i])_{i=1\cdots n}$ with the gluing diffeomorphisms $id, f_1, \cdots, f_n$. Obviously the manifold $\mathcal{M}(f_1, f_2, \cdots, f_n)$ is the manifold defined by the Heegaard data $(\mathcal{L}, f_n \circ f_{n-1} \circ \cdots \circ f_1)$, but it is also obvious that this manifold is that defined by the surgery data R. We will say that these surgery and Heegaard presentation of the same manifold are "related" description of $\mathcal{M}$.

4.2 Invariants associated to Heegaard diagrams and Lattice q-gauge theory

Our principal aim in this section is to prove the following theorem:

Proposition 10 (Invariants of three manifolds and Heegaard diagrams) *Let $(\mathcal{T}_g, (x_j)_{j=1,\ldots,g}, (y_j)_{j=1,\ldots,g}, (z_j)_{j=1,\ldots,g})$ be an augmented Heegaard diagram associated to a manifold $\mathcal{M}$ then the expectation value :*

$$\mathcal{J}_{\mathcal{M}} = \frac{< \prod_{i=1}^{g} \delta_{y_i} \prod_{i=1}^{g} \delta_{x_i} >_{q-YM(\mathcal{T}_g)}}{< \prod_{i=1}^{g} \delta_{z_i} \prod_{i=1}^{g} \delta_{x_i} >_{q-YM(\mathcal{T}_g)}} \tag{98}$$

is an invariant of the manifold $\mathcal{M}$. Moreover this value is equal to the Reshetikhin-Turaev invariant associated to the manifold $\mathcal{M}$.

The normalization by the expectation value associated to the sphere is chosen to obtain an $3-$manifold invariant equal to 1 for the sphere. **Remark 1:** The latter definition of the correlation function is in fact very natural from the general construction of q-gauge theory. Indeed putting a delta function associated to a plaquette P corresponds to imposing that any ribbon, i.e. holonomy defined in terms of the gauge fields algebra, can be displaced through P without torsion and without changing the expectation value. So, adding to the projector associated to the surface some delta functions corresponding to the x_is, i.e. canonical curves of the interior handlebody, and, at a future time, the delta functions of the y_is, i.e. canonical curves of the exterior handlebody, allows us to displace any curve through a handle of any of the two Heegaard components.This is exactly what we want to do in the framework of Chern-Simons theory.
Remark 2: Using the properties

$$\begin{aligned}&(\frac{\delta_C}{\sum_\alpha [d_\alpha]^2})^2 = (\frac{\delta_C}{\sum_\alpha [d_\alpha]^2}) \\ &(\frac{\delta_{C_1}}{\sum_\alpha [d_\alpha]^2})(\frac{\delta_{C_2}}{\sum_\alpha [d_\alpha]^2}) = (\frac{\delta_{C_1}}{\sum_\alpha [d_\alpha]^2})(\frac{\delta_{C_1\#C_2}}{\sum_\alpha [d_\alpha]^2})\end{aligned} \tag{99}$$

we can replace easily the correlation function by one where we put all Lickorish generators rather than the canonical curves only. This fact will be useful in the next section.

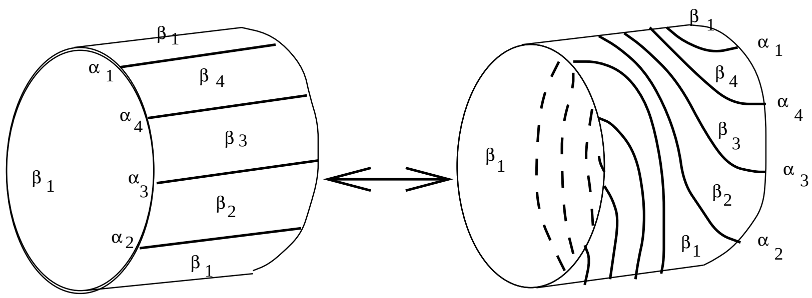
We are going to proove the last theorem through two lemmas describing some properties of this invariant.

Lemma 5 *The expectation value* $< \prod_{i=1}^g \delta_{y_i} \prod_{i=1}^g \delta_{x_i} >_{q-YM(\mathcal{T}_g)}$ *associated to a Heegaard diagram* $(\mathcal{T}_g, (x_j)_{j=1,\dots,g}, (y_j)_{j=1,\dots,g})$ *of a manifold* $\mathcal{M}$ *is invariant under any Singer's move applied to the diagram.*

Proof:
Trivial moves:
it is a fact already established that the expectation value is invariant under any isotopic deformation of any curve in the surface, simply because of the flatness condition.
We have the formula $\delta_C = \delta_{C^{-1}}$, which is exactly the second trivial move.
Handlebodies Diffeomorphisms:
The flatness condition implies trivially the following property for any curves C_1, C_2:

$$\delta_{C_1}\delta_{C_2} = \delta_{C_1}\delta_{C_1\#C_2} \tag{100}$$

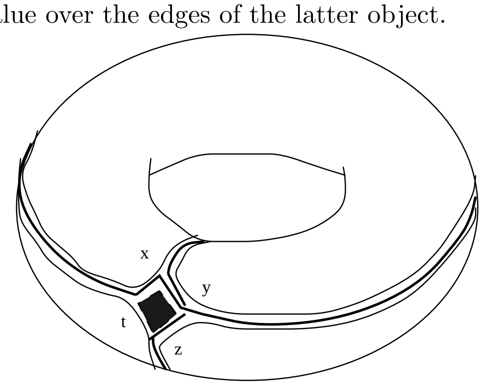
moreover the δ_{e_i}s (resp.δ_{f_i}s) commute one with the others. We then obtain the invariance under the first type 1 move.
It is easy to see, with the expression of the block element $\mathcal{A}_{free}$, that we can do the following replacement along any curve x_i.:



which implies the invariance under the second type 1 move.

$g \to g+1$ **moves:**

Let us consider a Heegaard diagram with one handle with its $a-$ and $b-$cycles. We cut the surface along a certain $2-$cell to obtain a torus with a puncture on which are drawn the two cycles as in the following figure. We choose the minimal fat graph describing this object to describe the partial integration of the expectation value over the edges of the latter object.



we obtain easily:

$$
\begin{aligned}
&\int dh(U_{[y,t]})dh(U_{[x,z]})\delta_{[y,t,x,z,t,y,z,x]}\delta_{[x,y,z,x]}\delta_{[x,y,t,x]} = \\
&= \int dh(U_{[y,t]})dh(U_{[x,z]})\delta_{[y,t,x,z,t]}\delta_{[x,y,z,x]}\delta_{[x,y,t,x]} \\
&= \int dh(U_{[y,t]})dh(U_{[x,z]})\delta_{[t,x,z]}\delta_{[x,y,z,x]}\delta_{[x,y,t,x]} \\
&= \int dh(U_{[y,t]})dh(U_{[x,z]})\delta_{[t,x,y,z]}\delta_{[x,y,z,x]}\delta_{[x,y,t,x]} \\
&= \delta_{[t,x,y,z]}
\end{aligned}
$$

the last line is easily obtained by using the property that the integration just "pick" the zero component associated to a link. The latter result establishes the invariance under the type 2 Singer move. This ends the proof of the lemma

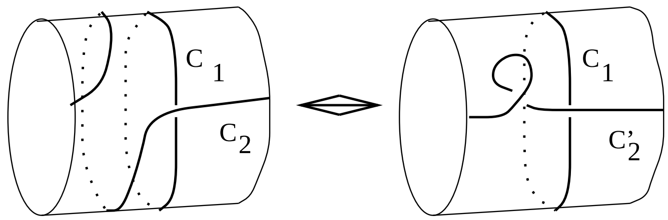
and shows that the expectation value is an invariant of the manifold $\mathcal{M}$. □

Lemma 6 *For any augmented Heegaard diagram* $(\mathcal{L}, (x_i)_{i=1\cdots g}, (y_i)_{i=1\cdots g}, (z_i)_{i=1\cdots g})$ *describing a manifold* $\mathcal{M}$ *there exists a framed link* L *which is a surgery data describing the same manifold* $\mathcal{M}$ *and verifying:*

$$\frac{< \prod_{i=1}^{g} \delta_{y_i} \prod_{i=1}^{g} \delta_{x_i} >_{q-YM(\mathcal{A})}}{< \prod_{i=1}^{g} \delta_{z_i} \prod_{i=1}^{g} \delta_{x_i} >_{q-YM(\mathcal{A})}} = RT(\mathcal{M}) \tag{101}$$

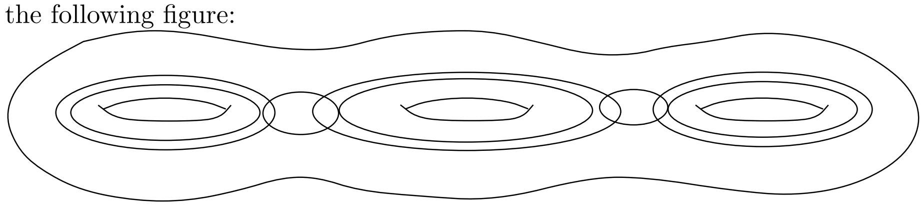
where RT *is the Reshetikhin Turaev invariant of the manifold computed from* L.

Proof: The trick already used in the proof of the invariance under the second Singer move can be used also here. We first use a natural property of delta functions that can be described by the following figure :



$$\delta_{C_2}\delta_{C_1} = \delta_{C'_2}\delta_{C_1} \tag{102}$$

to transform the correlation function in a new one $< (\sum_{\alpha_1,\cdots,\alpha_g}(\prod_i [d_{\alpha_i}])W((R_i,\alpha_i)_{i=1\cdots g}) \prod_{i=1}^{g} \delta_{x_i} >_{q-YM(\mathcal{L})}$ where the R_is are ribbons glued on the surface with the same framings and knotted in S^3 in the same way as the y_is but with a support now included in the area described in the following figure:



Using now the usual flatness property (66) we can deform again the latter knot to put its crossings in the "discs", the rest of the knot being composed of parallel strands along handles zones.

Then we are able to do the same calculus as in the verification of the invariance under type 2 Singer's move. The integration "picks" again the zero component on each segment of the skeleton. We then obtain the equality:

$$< \prod_i \delta_{y_i} \prod_i \delta_{x_i} >_{q-YM(\mathcal{A})} = \sum_{\alpha_1,\cdots,\alpha_g} (\prod_i [d_{\alpha_i}])(\prod_j I_{disc_j}) \tag{103}$$

with I_{disc_j} being the invariant associated, by our construction, to the knot contained in the j-th disc, placed on the sphere S^2 and with four coupons picking

the zero component on the boundary of the disc.
It is then easy to see, using the equivalence already established in section (3) between our invariant on the sphere and the Reshetikhin invariant of link , that the quantity I_{disc_j} is exactly the Reshetikhin invariant associated to this framed link with coupons.
Now the proof can be achieved by establishing, using the Reshetikhin-Turaev framework, that the latter data is a surgery data of the manifold $\mathcal{M}$. Let us first recall that, using the "related" surgery and Heegaard descriptions, we can replace the set of curves y_i describing the manifold $\mathcal{M}$ by a link composed of the curves z_i associated to the Heegaard description of S^3 placed at a time t and the curves $R_i \times t_i$ (with $t_i \leq t$) associated to the composition of Dehn twists describing the Heegaard gluing diffeomorphism encoded in the y_is. If we compute, with the notations of Reshetikhin and Turaev in [10], the invariant associated to the framed link described before, with an insertion of two "coupons" for each handle picking the zero component, we obtain easily that this invariant is equal to the invariant associated to the link $L = \cup_i R_i \times t_i$ only. This property uses trivially the fact that :

$$\sum_{\alpha,{\alpha'}_1,{\alpha'}_n} tr_{V_\alpha}(\overset{\alpha}{\mu}\, \phi^{\alpha_n \alpha_{n-1}}_{{\alpha'}_{n-1}} \cdots \phi^{\alpha_2\alpha_1}_{{\alpha'}_1} \phi_0^{{\alpha'}_1 \alpha} \psi^0_{{\alpha'}_1 \alpha} \psi^{{\alpha'}_1}_{\alpha_2\alpha_1} \cdots \psi^{{\alpha'}_{n-1}}_{\alpha_n\alpha_{n-1}}) = id_{V_{\alpha_1}} \otimes \cdots \otimes id_{V_{\alpha_n}}$$

We then obtain:

$$\frac{< \prod_{i=1}^g \delta_{y_i} \prod_{i=1}^g \delta_{x_i} >_{q-YM(\mathcal{L})}}{< \prod_{i=1}^g \delta_{z_i} \prod_{i=1}^g \delta_{x_i} >_{q-YM(\mathcal{L})}} = \sum_{(\alpha)} (\prod_i [d_{\alpha_i}]) RT((R_i, \alpha_i)_{i=1\cdots n})$$

Using now the celebrated result of [10] this object is a non trivial invariant of the 3-manifold $\mathcal{M}$, it is the Reshetikhin-Turaev's invariant of three manifolds.
$\square$

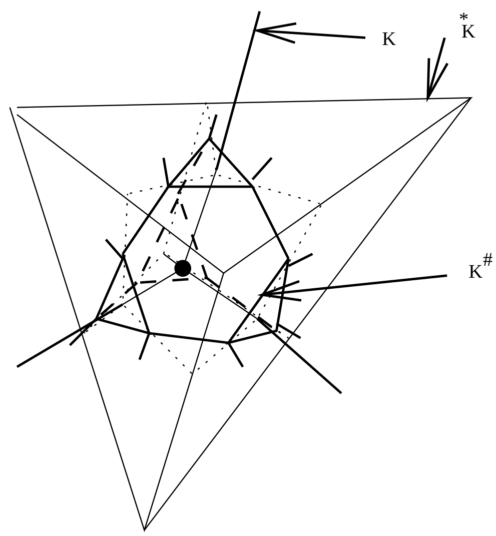
4.3 Chern-Simons theory on a lattice and Three dimensional Lattice q-gauge theory

We will define here a three dimensional gauge theory which extends in some sense the previous construction on a surface. The definition of this theory is based on a choice of a simplicial presentation of the manifold which exhibits naturally a canonical decomposition of the manifold. Let us consider a 3-manifold $\mathcal{M}$ given by a complex K. We impose here that all vertices of K are tetravalent. We will denote $K^\star$ the dual complex of K. We will denote as before $A_0^i, A_1^j, A_2^k, A_3^l$ the $0-,1-,2-,3-$simplexes of K , $A_0^{\star i}, A_1^{\star j}, A_2^{\star k}, A_3^{\star l}$ the $0-,1-,2-,3-$simplexes of $K^\star$ and B_1^j (resp.$B_1^{\star j}$) the middle of the A_1^j (resp. $A_1^{\star j}$).

Definition 15 (canonical thickening of a graph) *Let us define another tetravalent complex $K^\#$ build up from the previous one as follows: A couple of B_1^j and*

$B_1^{\star j}$ are said to be a couple of neighbours if B_1^j is the middle of an edge of a certain A_2^k and $B_1^{\star j}$ is in the middle of this A_2^k. We denote by J the set of $1-$simplexes defined by the set of couples of neighbour points. We now define the $0-$simplexes of $K^{\#}$ to be the middles of the elements of J. The $1-$simplexes of $K^{\#}$ are then given by the set of couples of $0-$ simplexes corresponding to elements of J having one vertex in common, if this vertex is a B_1^j (resp. a $B_1^{\star j}$) then this $1-$simplex is said "of type K" (resp. "of type $K^\star$"). Now the $2-$simplexes are defined to be of three types: one $2-$simplex is associated to each closed curve formed by type K $1-$simplexes only, one to each closed curve formed by type $K^\star$ $1-$simplexes only, and one to each closed curve formed alternatively by type K and type $K^\star$ $1-$simplexes. We will refer us to "the $e_i s$" , "the $f_j s$", and "the Ps" to denote respectively these three types of $2-$simplexes. Finally the $3-$simplexes are defined in an obvious way by considering each connected region around the vertices of K and $K^\star$.

We will denote by $K_0^{\#}, K_1^{\#}, K_2^{\#}, K_3^{\#}$ the sets of $0-, 1-, 2-, 3-$simplexes respectively. A piece of this new complex is shown in the following figure:



The graph $K^{\#}$ build up from any triangulation K describing a manifold $\mathcal{M}$ owns the following properties:

Lemma 7 *If we denote by $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) the region defined by the set of $3-$cells associated to vertices of K (resp.$K^\star$) and by $\mathcal{L}$ the surface defined by the set of Ps.The decomposition: $\mathcal{M} = \mathcal{R}_1 + \mathcal{L} + \mathcal{R}_2$ is a canonical decomposition and $K^{\#}$ is homeomorphic to K. The set formed by the elements of $K_0^{\#}$,the elements of $K_1^{\#}$ and all Ps forms the complex L associated to the triangulation of the canonical surface L(for this reason these sets of $0-, 1-$ and $2-$simplexes will be also denoted respectively by L_0, L_1, L_2)*

The set of canonical $2-$cells of $\mathcal{R}_1$ (resp $\mathcal{R}_2$) is a subset of the e_is (resp.the f_js).

Proof: This decomposition is equivalent to the Heegaard decomposition "derived" from the complex K. $\square$

Definition 16 (3-dimensional lattice q-gauge theory) *As a consequence of the property that $K_0^{\#} = L_0$ and $K_1^{\#} = L_1$, we can define as before the exchange algebra associated to the elements of $K_1^{\#}$ by imposing the coaction of the gauge symmetry algebra at each element of $K_0^{\#}$ and by choosing a cilium order on the surface. We can define as before the Wilson loops attached to each closed path formed by elements of $K_1^{\#}$ (i.e. drawn on L) and delta functions associated to each $2-$cell. In fact we define the Yang-Mills weight associated to a 2-cell P of area A_P to be:*

$$\delta_P^{\beta} = \sum_{\alpha \in Phys(A)} [d_\alpha] e^{-\frac{A_P C_\alpha}{2\beta}} \overset{\alpha}{W}_P \tag{104}$$

where C_α is the quadratic casimir of the representation α and β is a coupling constant of the Yang-Mills theory. We define the expectation value associated to any element $\mathcal{A}$ of Λ^{inv} in the 3 dimensional q-Yang Mills theory to be:

$$< \mathcal{A} >_{\mathcal{M}} := \int \prod_{l \in K_1^{\#}} dh(U_l) (\prod_j \delta_{f_j}^{\beta}) (\prod_{P \in L_2} \delta_P^{\beta}) \ \mathcal{A} \ (\prod_i \delta_{e_i}^{\beta}) \tag{105}$$

in the limit $q \to 1$ this theory becomes the well known Yang-Mills theory on a lattice associated to a manifold $\mathcal{M}$.

Proposition 11 *Let L be a link drawn on the 1-skeleton $K_1^{\#}$ of $\mathcal{M}$. Using again the properties of the complex K, L is in fact drawn on the canonical surface and we can define W_L in the framework defined in this article. The correlation function associated to L in the limit $\beta \to 0$ is then*

$$lim_{\beta \to \infty} \frac{< W_L >_{\mathcal{M}}}{< 1 >_{\mathcal{M}}} = \frac{RT(\mathcal{M}, L)}{RT(\mathcal{M})} \tag{106}$$

this formula can be considered as a description of Reshetikhin-Turaev invariants, i.e. of Chern-Simons invariants in term of a well defined lattice gauge theory and a definition of the Witten's path integral formulas.

Proof:
The expectation value is simply the same as that introduced in the last subsection but with a very special Heegaard decomposition where the gluing diffeomorphism is simply the identity.
$\square$

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